VCAA 2019 Mathematical Methods
Examination 2 Provisional Solutions

TWM Publications

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SECTION A – Multiple-choice questions

Question 1 (B)

The period and range of $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3\sin\left(\frac{2x}{5}\right) - 2$ are

$$\text{period} = \frac{2\pi}{2/5} = 5\pi \quad \text{and} \quad \text{ran}(f) = \big[-2-3,\ -2+3\big] = [-5,\ 1].$$

Question 2 (B)

The equation $x^2 + 2x - k = 0$ has two real solutions for x provided

$$\Delta = 2^2 - 4(1)(-k) > 0 \implies k \in (-1,\ \infty).$$

Question 3 (E)

The average rate of change of $f : \mathbb{R} \setminus \{4\} \to \mathbb{R},\ f(x) = \frac{a}{x-4}$, where $a > 0$, from $x = 6$ to $x = 8$ is given by

$$\overline{f}' = \frac{f(8) - f(6)}{8 - 6} = \frac{-a}{8}.$$

Question 4 (C)

Using a CAS,

$$\int_0^{\pi/6} \Big(a\sin(x) + b\cos(x)\Big)\mathrm{d}x = \frac{(2-\sqrt{3})a + b}{2}.$$

Question 5 (C)
For $f'(x) = 3x^2 - 2x$, $f(4) = 0$, we have

$$f(x) = \int \big(3x^2 - 2x\big)\mathrm{d}x = x^3 - x^2 + C, \quad C \in \mathbb{R}.$$

Then, using the condition gives $c = -48$, and so

$$f(x) = x^3 - x^2 - 48.$$

Question 6 (A)
The volume of the box is given by

$$V(x) = x(80 - 2x)(50 - 2x),$$

and so V will be maximum when

$$V'(x) = 12x^2 - 520x + 4000 = 0 \implies x = 10.$$

Question 7 (D)
First, to find a, we require

$$a + 3a + 5a + 7a = 16 = 1 \implies a = \frac{1}{16}.$$

Hence, the mean of X is

$$\mathrm{E}(X) = \left(0 \times \frac{1}{16}\right) + \left(1 \times \frac{3}{16}\right) + \left(2 \times \frac{5}{16}\right) + \left(3 \times \frac{7}{16}\right) = \frac{17}{8}.$$

Question 8 (C)
Let $X \sim \mathrm{Bi}(80,\ 0.9)$ be the number of times the archer hits the target. Then,

$$\Pr(X = 74 \,|\, X \geq 70) = \frac{\Pr(X = 74)}{\Pr(X \geq 70)} = \frac{0.123535...}{0.826616...} = 0.1494 \quad \text{(4DP)}.$$

Question 9 (E)
The transformation gives

$$\frac{1}{2}a - \frac{1}{2} = 0 \quad \text{and} \quad -2b - 2 = 0 \implies a = 1 \quad \text{and} \quad b = -1.$$

Question 10 (D)

For $f : \mathbb{R} \to \mathbb{R},\ f(x) = x + \sin(x)$, we have

$$f'(x) = 1 + \cos(x) \geq 0$$

since $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$.

Question 11 (A)

For A and B to be independent events, so must A' and B, so

$$\Pr(B) = \Pr(B \,|\, A) = m \quad \text{and} \quad \Pr(B) = \Pr(B \,|\, A') = n \quad \Longrightarrow \quad m = n.$$

Question 12 (E)

Using integral properties,

$$\int_1^2 \big(f(x) + x\big)\mathrm{d}x = \int_1^4 f(x)\,\mathrm{d}x - \int_2^4 f(x)\,\mathrm{d}x + \int_1^2 x\,\mathrm{d}x = 4 - (-2) + \frac{3}{2} = \frac{15}{2}.$$

Question 13 (C)

The graph of $h(x) = f\left(\dfrac{x}{2}\right) + 5$ is obtained from the graph of f by a dilation by factor 2 from the y-axis, followed by a translation of 5 units in the positive y-direction. Thus, the graph of h must pass through

$$(-2 \times 2,\ 7 + 5) = (-4,\ 12).$$

Question 14 (B)

Let $W \sim \mathcal{N}(200,\, \sigma^2)$ be the weight of a randomly selected packet of lollies. Then,

$$0.97 = \Pr(W > 190) = \Pr\left(Z > \frac{190 - 200}{\sigma}\right)$$

$$\Longrightarrow \quad \frac{190 - 200}{\sigma} = -1.88079... \Longrightarrow \sigma = 5.3 \text{ g} \quad \text{(1DP)}.$$

Question 15 (A)

Here, $f : [2,\, \infty) \to \mathbb{R},\ f(x) = x^2 - 4x + 2$, and g is the inverse function of f. With $f(5) = 7$, by the inverse function theorem,

$$g'(7) = \frac{1}{f'(5)} = \frac{1}{2 \times 5 - 4} = \frac{1}{6}.$$

Question 16 (A)

The given graph of f presumably has $f'(x) < 0$ for $x \in (-\infty,\, 5) \setminus \{0\}$, $f'(0) = f'(5) = 0$ and $f'(x) > 0$ for $x \in (5,\, \infty)$, which is only satisfied by option A.

Question 17 (D)

With n marbles, k of which are red, the probability of drawing two marbles of the same colour, without replacement, is

$$\Pr(E) = \left(\frac{k}{n} \times \frac{k-1}{n-1}\right) + \left(\frac{n-k}{n} \times \frac{n-k-1}{n-1}\right) = \frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}.$$

Question 18 (D)

The rule of p is given by

$$p(x) = \begin{cases} 2ax + 2a, & -a \le x \le 0 \\ \dfrac{b-2a}{b}x + 2a, & 0 < x \le b \end{cases}.$$

Now, since f is a probability density function, we must have

$$\int_{-a}^{b} p(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \frac{1}{b-(-a)}\int_{-a}^{b} p(x)\,\mathrm{d}x = \frac{3}{4}$$

$$\implies \frac{1}{2}a(2a) + \frac{1}{2}b(2a+b) = 1 \quad \text{and} \quad \frac{1}{a+b} = \frac{3}{4}$$

Solving these simultaneously gives

$$a = \frac{\sqrt{2}}{3} \quad \text{and} \quad b = \frac{4-\sqrt{2}}{3}.$$

Thus,

$$\Pr(X > 0) = \frac{1}{2}b(2a+b) = \frac{7}{9}.$$

Question 19 (E)

If $\tan(\alpha) = d$, $d > 0$, $0 < \alpha < \frac{\pi}{2}$, then

$$\tan(2x) = d \implies 2x = \alpha + \pi k,$$

where k takes on appropriate integers. Thus, for $0 < x < \frac{5\pi}{4}$, we have

$$\tan(2x) = d \implies x = \frac{\alpha}{2},\ \frac{\alpha}{2} + \frac{\pi}{2},\ \frac{\alpha}{2} + \pi,$$

and so the sum of the solutions is

$$s = \frac{3(\pi + \alpha)}{2}.$$

Question 20 (D)

Using the change of base law,

$$\log_x(y) + \log_y(z) = \frac{\log_y(y)}{\log_y(x)} + \frac{\log_z(z)}{\log_z(y)} = \frac{1}{\log_y(x)} + \frac{1}{\log_z(y)}.$$

SECTION B

Question 1a (1 mark)
For $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 e^{-x^2}$, we have

$$f'(x) = (2x - 2x^3)e^{-x^2}.$$

Question 1b.i (1 mark)
For small $\epsilon > 0$, we have

$$f'(0 - \epsilon) < 0 \quad \text{and} \quad f'(0 + \epsilon) > 0,$$

so $(0, 0)$ is a local minimum.

Question 1b.ii (2 marks)
We have

$$f'(x) = 0 \implies x = 0,\ \pm 1,$$

and $x = \pm 1$ gives the maximum value of f since f is bounded. Hence

$$f_{\max} = f(\pm 1) = \frac{1}{e}.$$

Question 1b.iii (1 mark)
For $f(x) + d < 0$ for all $x \in \mathbb{R}$, it is sufficient that

$$f(\pm 1) + d < 0 \implies d \in \left(-\infty,\ \frac{-1}{e}\right).$$

Question 1c.i (1 mark)
We already know that $f'(-1) = 0$ and $f(-1) = 1/e$, so the tangent is given by

$$y = \frac{1}{e}.$$

Question 1c.ii (2 marks)
The area of the enclosed region is

$$A = \int_{-1}^{1} \left(\frac{1}{e} - f(x)\right) \mathrm{d}x = 0.3568 \text{ units}^2 \quad \text{(4DP)}.$$

Question 1d (3 marks)
The distance d from M to $(0, e)$ is given by

$$d(m) = \sqrt{(m-0)^2 + (f(m)-e)^2}.$$

For the minimum distance,

$$d'(m) = 0 \implies m = 0.783 \quad \text{(3DP)},$$

and so the minimum distance is

$$d_{\text{min}} = d(0.783034...) = 2.511 \text{ units} \quad \text{(3DP)}.$$

Question 2a (1 mark)
For $y = \dfrac{3x(x-30)^2}{2000}$, we have

$$\frac{dy}{dx} = \frac{9}{2000}(x-30)(x-10).$$

Question 2b (1 mark)
We have

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{9x}{1000} - \frac{9}{50} = 0 \implies x = 20.$$

Thus, noting that $\dfrac{dy}{dx}$ is suitably defined for $x \in (0,\ 30)$, the *gradient*, $\dfrac{dy}{dx}$, is strictly decreasing for $x \in (0,\ 20]$.

If instead we assume taking a one-sided gradient at $x = 20$ is appropriate, then one can also argue for the answer $[0,\ 20]$. Note that the inclusion of $x = 20$ is deliberate however.

Question 2c (1 mark)
The cable can be modelled by

$$y_c(x) = \frac{3x(x-30)^2}{2000} + 3 = \frac{3x^3}{2000} - \frac{9x^2}{100} + \frac{27x}{20} + 3, \quad x \in [a,\ 30].$$

Question 2d (3 marks)
We wish to solve

$$y_c'(x) = \frac{y(30) - y(10)}{30 - 10} \implies \frac{9x^2}{2000} - \frac{9x}{5} + \frac{27}{20} = \frac{-3}{10} \implies x = \frac{60 \pm 10\sqrt{3}}{3}.$$

Question 2e.i (1 mark)

The gradient of the cable at A is given by

$$m_A = y_c'(a) = \frac{9a^2}{2000} - \frac{9a}{50} + \frac{27}{20}.$$

Alternatively, using the straight section,

$$m_A = \frac{10 - y_c(a)}{0 - a} = \frac{3a^3 - 180a^2 + 2700a - 14000}{2000a}.$$

Question 2e.ii (3 marks)

The cable must be 'smooth', so we have

$$\frac{3a^3 - 180a^2 + 2700a - 14000}{2000a} = \frac{9a^2}{2000} - \frac{9a}{50} + \frac{27}{20} \implies a = 11.12 \quad \text{(2DP)}.$$

Hence, we have $b = y_c(11.1157...) = 8.95$ (2DP), and so

$$A = (11.12,\ 8.95) \quad \text{(2DP)}.$$

Question 2e.iii (1 mark)

Hence, the gradient of the cable at A is

$$y_c'(11.1157...) = -0.1 \quad \text{(1DP)}.$$

Question 3a (1 mark)

From the graph, the period of $f : [0,\ \infty) \to \mathbb{R},\ f(t) = \sin\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{6}\right)$ is 12.

Question 3b (1 mark)

Setting $f(t) = 0$ for $t \in [0,\ 6]$ gives

$$t = 0,\ 4,\ 6.$$

Question 3c (1 mark)

Using a CAS,

$$f_{\max} = 1.76 \quad \text{(2DP)}.$$

Question 3d (2 marks)

The area bounded by the graph of f and the horizontal axis is

$$A = \int_0^4 f(t)\,\mathrm{d}t - \int_4^6 f(t)\,\mathrm{d}t = \frac{15}{\pi} \text{ units}^2.$$

Question 3e (2 marks)

Through some experimentation, noting the importance of the domain of f, the family of functions g can be obtained from f via a reflection in the y-axis, followed by a translation of 6, 18, 30, ... units in the positive x-direction. This construction gives

$$a = -1 \quad b = 1, \quad c = 6(2k-1), \quad k \in \mathbb{Z}^+ \quad \text{and} \quad d = 0.$$

Alternatively, we can obtain the functions g via a reflection in the x-axis, followed by a translation of 6, 18, 30, ... units in the negative x-direction. This construction gives

$$a = 1 \quad b = -1, \quad c = -6(2k-1), \quad k \in \mathbb{Z}^+ \quad \text{and} \quad d = 0.$$

Note, there are many alternatives that we haven't considered. In fact, there are infinitely many possible sets, so we believe VCAA will be looking for just one possible set of values, ie. one of the above where $k = 1$ for example.

Question 3f (2 marks)

Using our result from *Q3d*, we have

$$12k = 2 \times \frac{15}{\pi} \implies k = \frac{5}{2\pi}.$$

Question 4a (2 marks)

Here, $X \sim f$, where $f(x) = \dfrac{4}{625}(5x^3 - x^4)$ for $x \in [0, 5]$. Thus, the mean life span of a Lorenz birdwing butterfly is

$$\mathrm{E}(X) = \int_0^5 f(x)\,\mathrm{d}x = \frac{10}{3} \text{ weeks.}$$

Question 4b (2 marks)

We expect

$$80\int_2^5 f(x)\,\mathrm{d}x = 73 \quad \text{(0DP)}$$

of the 80 to live longer than two weeks.

Question 4c (2 marks)

We have

$$\Pr(X \geq 4 \,|\, X \geq 2) = \frac{\displaystyle\int_4^5 f(x)\,\mathrm{d}x}{\displaystyle\int_2^5 f(x)\,\mathrm{d}x} = 0.2878 \quad \text{(4DP).}$$

Question 4d (1 mark)

Let $W \sim \mathcal{N}(14.1,\ 2.1^2)$ denote wingspan. Then,

$$\Pr(16 \leq W \leq 18) = 0.1512 \quad \text{(4DP).}$$

Question 4e (1 mark)
Here, we have

$$\Pr(W < b) = 0.05 \implies b = 10.6 \text{ cm} \quad (1\text{DP}).$$

Question 4f.i (1 mark)
Let $L \sim \text{Bi}(36, 0.0527)$ be the number of very large butterflies from 36. Then,

$$\Pr(L \geq 3) = 0.2949 \quad (4\text{DP}).$$

Question 4f.ii (2 marks)
Here, we require that

$$\Pr(W \geq n) < 0.01 \implies n_{\min} = 7,$$

by trial and error.

Question 4f.iii (2 marks)
Where $\widehat{P} = \dfrac{L}{36}$, we have

$$\text{E}(\widehat{P}) = 0.0527 \quad \text{and} \quad \text{sd}(\widehat{P}) = \sqrt{\frac{0.0527 \times 0.9473}{36}} = 0.0372 \quad (\text{both 4DP}).$$

Question 4f.iv (2 marks)
We have

$$\Pr(0.0527 - 0.0372 \leq \widehat{P} \leq 0.0527 + 0.0372) = \Pr(1 \leq L \leq 3) = 0.7380 \quad (4\text{DP}).$$

Question 4g (2 marks)
The sample proportion used to calculate the interval is

$$\widehat{p} = \frac{0.0234 + 0.0866}{2} = 0.055,$$

and hence we have

$$1.96\sqrt{\frac{0.055 \times 0.945}{n}} = \frac{0.0866 - 0.0234}{2} \implies n = 200.$$

Question 5a (1 mark)
Here, $f : \mathbb{R} \to \mathbb{R},\ f(x) = 1 - x^3$, so the tangent to the graph of f at $x = a$ is

$$L(x) = f'(a)(x - a) + f(a) = -3a^2x + 2a^3 + 1.$$

Question 5b (1 mark)

Hence,

$$L(x) = 0 \implies x = \frac{2a^3 + 1}{3a^2}.$$

Question 5c (2 marks)

Then,

$$L(x) = f(x) \implies x = -2a.$$

Question 5d (3 marks)

The area of the shaded region is given by

$$A(a) = \int_{-2a}^{1} \Big(L(x) - f(x) \Big) \mathrm{d}x + \int_{1}^{\frac{2a^3+1}{3a^2}} L(x) \, \mathrm{d}x = \frac{20a^4}{3} + \frac{2a}{3} + \frac{1}{6a^2} - \frac{3}{4}.$$

Question 5e (2 marks)

For the minimum area, we have

$$A'(a) = \frac{80a^3}{3} - \frac{1}{3a^2} + \frac{2}{3} = 0 \implies a = 10^{-1/3}.$$

Question 5f (2 marks)

By symmetry of f and f^{-1} about the line $y = x$, we know that

$$b = f(10^{-1/3}) = \frac{9}{10}$$

Question 5g (1 mark)

The acute angle of interest is formed by the lines $y = 3 - 3x$ and $x = 1$. Thus,

$$\theta = \arctan\left(\frac{1}{3}\right).$$