Vector Calculus

2008년 9월 4일

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제 1 장

The geometry of Euclidean Space

We consider the basic operations on vectors in 3 and 3 dim. space: vector addition, scalar multiplication, dot an cross product. In section 5 we generalize these notions to n dim'l space.

제 1 절　Vectors in 2, 3 dim space

A point P is represented by ordered pairs of real numbers (a_1, a_3) called **Cartesian coordinate)** of P.

a_1, a_2 are called x **coordinate**, y **coordinate** or x **component**, y **component** of (a_1, a_2). x**- axis**, y**- axis**. The point $(0, 0)$ is called the **origin** and denoted by O.

(1) The set of all real numbers is denoted by $\mathbb{R}$.

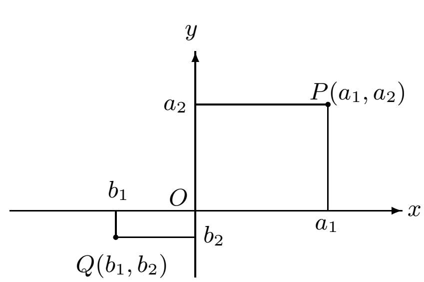


그림 1.1: Coordinate plane

(2) The set of all ordered pairs of real numbers (x, y) is denoted by $\mathbb{R}^2$.

(3) The set of all ordered triples of real numbers (x, y, z) is denoted by $\mathbb{R}^3$.

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$$

The planes in $\mathbb{R}^3$determined by $z = 0$, (resp. $x = 0$ and $y = 0$) are called

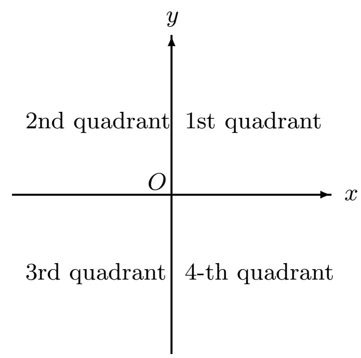


그림 1.2: quadrant

***xy*-plane, (resp. *yz*-plane, *zx*-plane)** These planes divides the space into eight parts: Each of them is called **octant**. If every component is positive, it is called **the first octant**.

Example 1.1. (1) The xz-plane is the set of all points with $y = 0$.

$$\{(x, y) \mid y = 0\}$$

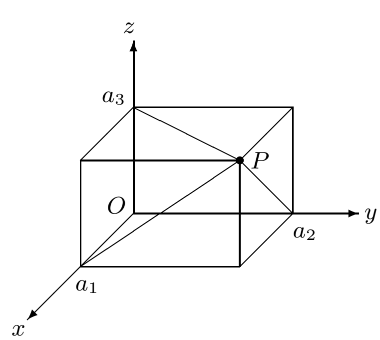


그림 1.3: point $P(a_1, a_2, a_3)$

(2) Similarly, the xy-plane is determined by $z = 0$.

$$\{(x, y, z) \mid z = 0\}$$

(3) x-axis is determined by

$$\begin{cases} y = 0 \\ z = 0 \end{cases}$$

or

$$\{(x, y, z) \mid y = 0, z = 0\}$$

Vector addition and scalar multiplication

The operation of addition can be extended to $\mathbb{R}^3$ **sum** Given two triples, $(a_1, a_2, a_3) + (b_1, b_2, b_3)$ we define

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

to be the **sum** of (a_1, a_2, a_3) and (b_1, b_2, b_3). $(0, 0, 0)$ is the **zero element**. $(-a_1, -a_2, -a_3)$, or $-(a_1, a_2, a_3)$ called **additive inverse or negative** of (a_1, a_2, a_3). We have commutative law and associate law.

(i) $(x, y, z) + (u, v, w) = (u, v, w) + (x, y, z)$ (commutative law)

(ii) $((x, y, z) + (u, v, w)) + (l, m, n)$
$= (x, y, z) + ((u, v, w) + (l, m, n))$ (associate law)

difference is defined as.

$$\begin{aligned}(a_1, a_2, a_3) - (b_1, b_2, b_3) &= (a_1, a_2, a_3) + (-(b_1, b_2, b_3)) \\ &= (a_1 - b_1, a_2 - b_2, a_3 - b_3)\end{aligned}$$

Example 1.2.

$$\begin{aligned}(6, 0, 2) + (-10, 3, 2) &= (-4, 3, 4) \\ (3, 0, 3) - (5, 0, -2) &= (-2, 0, 5) \\ (0, 0, 0) + (1, 3, 2) &= (1, 3, 2)\end{aligned}$$

For any real α, and (a_1, a_2, a_3) in $\mathbb{R}^3$ $\alpha(a_1, a_2, a_3)$ **scalar multiple** is defined as

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$$

Additions and scalar multiplication has the following properties:

(i) $(\alpha\beta)(x, y, z) = \alpha(\beta(x, y, z))$ (associate law)

(ii) $(\alpha + \beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributive law)

(iii) $\alpha((x, y, z) + (u, v, w)) = \alpha(x, y, z) + \alpha(u, v, w)$ (distributive law)

(iv) $\alpha(0, 0, 0) = (0, 0, 0)$ (property of 0)

(v) $0(x, y, z) = (0, 0, 0)$ (property of 0)

(vi) $1(x, y, z) = (x, y, z)$ (property of 1)

Example 1.3.

$$\begin{aligned} 3\,(6, -3, 2) &= (18, -9, 6) \\ 1\,(3, 5, -2) &= (3, 5, -2) \\ 0\,(1, 3, 2) &= (0, 0, 0) \\ (-2)\,(-2, 1, 3) &= (4, -2, -6) \end{aligned}$$

$$\begin{aligned} (x, y) + (u, v) &= (x + u, y + v) \\ \alpha(x, y) &= (\alpha x, \alpha y) \end{aligned}$$

Example 1.4. Show

(1) $(\alpha + \beta)(x, y) = \alpha(x, y) + \beta(x, y)$

(2) $\alpha((x, y) + (u, v)) = \alpha(x, y) + \alpha(u, v)$

sol. (1) LHS is

$$\begin{aligned}(\alpha+\beta)(x,y) &= ((\alpha+\beta)x, (\alpha+\beta)y) \\ &= (\alpha x+\beta x, \alpha y+\beta y) \\ &= (\alpha x, \alpha y)+(\beta x, \beta y) \\ &= \alpha(x,y)+\beta(x,y)\end{aligned}$$

(2) LHS is

$$\begin{aligned}\alpha((x,y)+(u,v)) &= \alpha(x+u, y+v) \\ &= (\alpha(x+u), \alpha(y+v)) \\ &= (\alpha x+\alpha u, \alpha y+\alpha v) \\ &= (\alpha x, \alpha y)+(\alpha u, \alpha v) \\ &= \alpha(x,y)+\alpha(u,v)\end{aligned}$$

Geometry of vectors

Define a **vector** to be a **directed line segment beginning at the origin**, i.e, a line segment with specified *magnitude* and *direction*, and initial point at the origin. Vectors are usually denoted by boldface such as $\mathbf{a}$ or $\vec{a}$. We associate with each vector $\mathbf{a}$ with each point (a_1, a_2, a_3) where $\mathbf{a}$ terminates. Conversely, we associate a vector $\mathbf{a}$ with a point (a_1, a_2, a_3) in the space. Thus, we identify $\mathbf{a}$ with (a_1, a_2, a_3) and write $\mathbf{a} = (a_1, a_2, a_3)$.

The elements in $\mathbb{R}^3$ are not only ordered triple of numbers, but are also regarded as vectors. We call a_1, a_2 and a_3 the **components** of $\mathbf{a}$. The triple $(0,0,0)$ is called **(zero vector)** denoted by $\mathbf{0}$ or $\vec{0}$.

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$. Geometrically, this means they have the same direction and magnitude.

Geomteric representation of vectors

See Figure 1.4. The directed line segment PQ from P to Q is denoted by $\overrightarrow{PQ}$. P and Q are called **tail** and **head** respectively. The vector with tail at origin is called **position vector**. If we move the vector in parallel, we regard

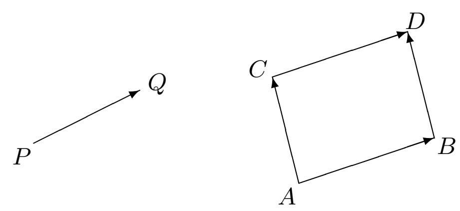


그림 1.4: vector

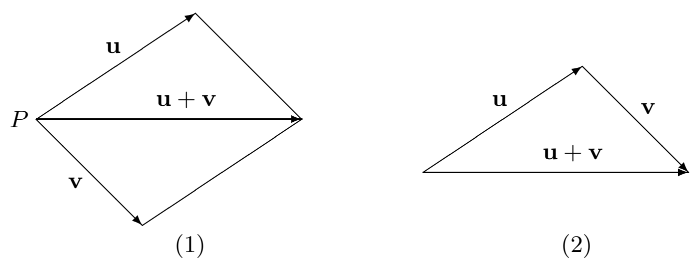


그림 1.5: sum of two vectors

it the same vector. In other words, a vector is determined by direction and magnitude. In Figure 1.4, $ABDC$ parallelogram, $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.

See figure 1.5 (1) If two vectors $\mathbf{u}$, $\mathbf{v}$ have same tail P, the vector with tail at P and head at opposite of P is defined as the sum of $\mathbf{u}$ and $\mathbf{v}$. denoted by $\mathbf{u} + \mathbf{v}$. Fig 1.5 (2)

If we insist vectors beginning at the origin, we say we have **bound vector**. If we allow vector to begin at other points we say **free vectors** or just **vectors**.

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative law)

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associate law)

Scalar multiple of a vector. For real s and vector $\mathbf{v}$, $s\mathbf{v}$ is the vector having

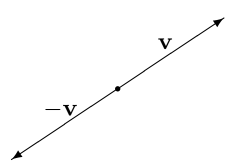


그림 1.6: $\mathbf{v}$ and $-\mathbf{v}$

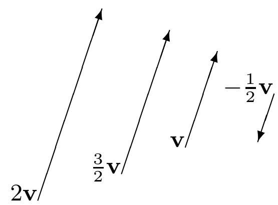


그림 1.7: scalar multiple of $\mathbf{v}$

magnitude $|s|$, having same direction as $\mathbf{v}$ when $s > 0$, opposite direction when $s < 0$. s is called **scalar** $s\mathbf{v}$ is the **scalar multiple** of $\mathbf{v}$ (Fig 1.11). The following hold:

(iii) $(st)\mathbf{u} = s(t\mathbf{u})$ (associative law)

(iv) $(s+t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ (distributive law)

(v) $s(\mathbf{u}+\mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ (distributive law)

(vi) $s\mathbf{0} = \mathbf{0}$ (0-vector)

(vii) $0\mathbf{u} = \mathbf{0}$ (0)

(viii) $1\mathbf{u} = \mathbf{u}$ (1)

Example 1.5. Show that $(-s)\mathbf{v} = -(s\mathbf{v})$ for any scalar s and vector $\mathbf{v}$.

Example 1.6 (3D)**.**

$$\mathbf{a} = (a_1, a_2, a_3)$$

$a_1,\ a_2,\ a_3$ are called x **component,** y **component,** z **component** of $\mathbf{a}$. As in figure 1.8 when $A = (a_1, a_2, a_3)$ shift the line segment OA by b_1 along x-axis, by b_2 along y-axis, b_3 along z-axis is denoted by BP. Then the coordinate of B is (b_1, b_2, b_3), P is $(a_1+b_1, a_2+b_2, a_3+b_3)$ and $OBPA$ is parallelogram. Hence

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OP}$$

In vector notation, we have

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3)$$

similarly

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, , \alpha a_3)$$

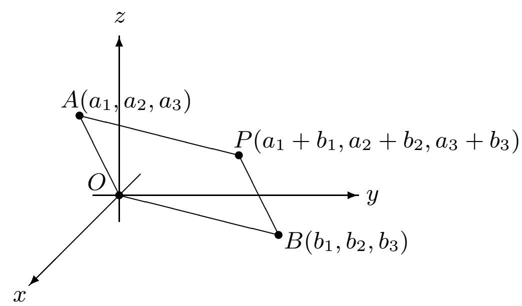


그림 1.8: Addition

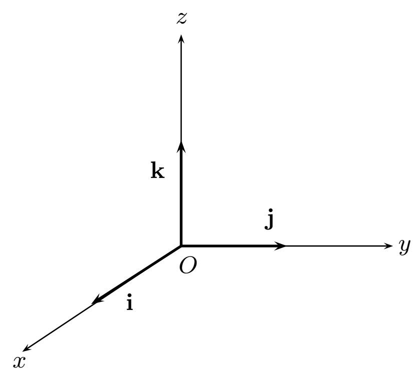


그림 1.9: standard basis vector

standard basis vectors

Definition 1.7. The following vectors **i**, **j**, **k** are called **(standard basis vector)** of $\mathbb{R}^3$ (Figure 1.13).

$$\mathbf{i} = (1, 0, 0),\ \ \mathbf{j} = (0, 1, 0),\ \ \mathbf{k} = (0, 0, 1)$$

Remark 1.8. (1) For a given $\mathbf{v} = (a_1, a_2, a_3)$

$$(a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1)$$

we write $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(2)

Example 1.9. Write the following using standard basis vectors.

(1) $\mathbf{v} = (-1/2, 3, 5)$

(2) Express $3\mathbf{a} - 2\mathbf{b}$ when $\mathbf{a} = (3, 5, 0),\ \mathbf{b} = (-4, 1, 1)$

(3) For two points $P(1, 4, 3),\ Q(4, 1, 2)$, express $\overrightarrow{PQ}$

(4) For three points $A(0, -1, 4),\ B(2, 4, 1),\ C(3, 0, 2)$, express

$$\frac{1}{2}\overrightarrow{OA} + \frac{1}{3}\overrightarrow{OB} + \frac{1}{6}\overrightarrow{OC}$$

sol. (1) $\mathbf{v} = (-1/2)\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

(2) $3\mathbf{a} - 2\mathbf{b} = 3(3\mathbf{i} + 5\mathbf{j}) - 2(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= (9 + 8)\mathbf{i} + (15 - 2)\mathbf{j} + (-2)\mathbf{k} = 17\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}$

(3) $\overrightarrow{PQ} = (4 - 1)\mathbf{i} + (1 - 4)\mathbf{j} + (2 - 3)\mathbf{k} = 3\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

(4) $(1/2)\overrightarrow{OA} + (1/3)\overrightarrow{OB} + (1/6)\overrightarrow{OC}$
$= (1/2)(-\mathbf{j} + 4\mathbf{k}) + (1/3)(2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + (1/6)(3\mathbf{i} + 2\mathbf{k})$
$= (7/6)\mathbf{i} + (5/6)\mathbf{j} + (8/3)\mathbf{k}$

Equation of lines(Point-Direction form)

$$\ell(t) = \mathbf{a} + t\mathbf{v}$$

The equation of the line ℓ through the tip of $\mathbf{a}$ and pointing in the direction of $\mathbf{v}$ is $\ell(t) = \mathbf{a} + t\mathbf{v}$ where t takes all real values. In coordinate form, we have

$$\begin{aligned} x &= x_1 + at, \\ y &= y_1 + bt, \\ z &= z_1 + ct, \end{aligned}$$

where $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{v} = (a, b, c)$.

Example 1.10. (1) Find equation of line through $(2, 1, 5)$ in the direction of $4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

(2) In what direction, the the line $x = 3t - 2, y = t - 1, z = 7t + 4$ points ?

sol. (1) $\mathbf{v} = (2, 1, 5) + t(4, -2, 5)$

(2) $(3, 1, 7) = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

Draw figure

Example 1.11. Does the two lines $(x, y, z) = (t, -6t + 1, 2t - 8)$ and $(3t + 1, 2t, 0)$ intersect ?

sol. If two line intersect, we have for two numbers t_1, t_2

$$(t_1, -6t_1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$$

no solution.

Two point form

We describe the equation of line through two points $\mathbf{a}, \mathbf{b}$.

Direction is $\mathbf{v} = \mathbf{b} - \mathbf{a}$. So by point -direction form

$$\ell(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

If $P = (x_1, y_1, z_1)$ is the tip of $\mathbf{a}$ and $Q = (x_2, y_2, z_2)$ is the tip of $\mathbf{b}$. then $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and hence the equation of line are

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \\ z &= z_1 + (z_2 - z_1)t \end{aligned}$$

Eliminating t, equation can be written as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Example 1.12. Find eq. of a line through $(2, 1, -3)$ and $(6, -1, -5)$.

Example 1.13. Find eq. of line segment between $(1, 1, -3)$ and $(2, -1, 0)$

sol. $0 \leq t \leq 1$

제 2 절 inner product, length, distance

Inner product

Definition 2.1. Given two vectors $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ we define

$$a_1b_1 + a_2b_2 + a_3b_3$$

to be the **(inner product)** of $\mathbf{u}$ and $\mathbf{v}$ and write $\mathbf{u} \cdot \mathbf{v}$.

Example 2.2. Let $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find

(1) $\mathbf{u} \cdot \mathbf{u}$

(2) $\mathbf{u} \cdot \mathbf{v}$

(3) $\mathbf{u} \cdot (\mathbf{u} - 3\mathbf{v})$

(4) $(3\mathbf{u} + 2\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

sol. (1) $\mathbf{u} \cdot \mathbf{u} = 4 + 9 + 1 = 14$

(2) $\mathbf{u} \cdot \mathbf{v} = 2 - 6 - 1 = -5$

(3) $\mathbf{u} \cdot (\mathbf{u} - 3\mathbf{v}) = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} - 9\mathbf{j} + 4\mathbf{k})$
$= -2 + 27 + 4 = 29$

(4) $(3\mathbf{u} + 2\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (8\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$
$= 8 + 25 + 2 = 35$

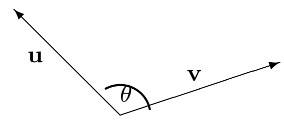


그림 1.10: Angle between two vectors

Proposition 2.3 (Properties of Inner Product)**.** *For vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *and scalar* α, *the following hold:*

(1) $\mathbf{u} \cdot \mathbf{u} \geq 0$ *(equality holds when* $\mathbf{u} = \mathbf{0}$*)*

(2) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(3) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(4) $(\alpha\mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v})$

(5) $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Proof. these can be proved easily. □

Example 2.4. For $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ Show the following.

(1) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(3) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(4) $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2)$

sol. We see

(1) $$\begin{aligned}(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} &= (\mathbf{u} + (-1)\mathbf{v})) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + ((-1)\mathbf{v}) \cdot \mathbf{w} \\ &= \mathbf{u} \cdot \mathbf{w} + (-1)\mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}\end{aligned}$$

(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(3) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = (\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(4) $$\begin{aligned}\|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 - 2\,\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2\end{aligned}$$

Length of vectors

The **length** of a vector $\mathbf{v} = (a_1, a_2, a_3)$ is

$$\sqrt{(a_1-0)^2 + (a_2-0)^2 + (a_3-0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

denoted by $\|\mathbf{v}\|$. Also,

$$\|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2}.$$

Proposition 2.5. *(1)* $\mathbf{u} = 0$ *iff* $\|\mathbf{u}\| = 0$.
(2) For scalar α, *it holds that* $\|\alpha\mathbf{u}\| = |\alpha|\,\|\mathbf{u}\|$.

Example 2.6. Find the length

(1) $\mathbf{u} = (3, 2, 1)$

(2) $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

(3) For $A(2, -1/3, -1)$, $B(8/3, 0, 1)$ $\overrightarrow{AB}$

sol. (1) $\|\mathbf{u}\| = \sqrt{9+4+1} = \sqrt{14}$
(2) $\|3\mathbf{i} - 4\mathbf{j} + \mathbf{k}\| = \sqrt{9+16+1} = \sqrt{26}$
(3) $\|\overrightarrow{AB}\| = \sqrt{(8/3-2)^2 + (0-(-1/3))^2 + (1-(-1))^2}$
$= \sqrt{4/9 + 1/9 + 4} = \sqrt{41}/3$

Definition 2.7. Vectors with norm 1 is called **(unit vector)** Any nonzero vector $\mathbf{u}$ can be made into a unit vector by $\mathbf{u}/\|\mathbf{u}\|$ This process is called to **normalize u**.[1]

Example 2.8. normalize.

(1) $\mathbf{i} + \mathbf{j} + \mathbf{k}$

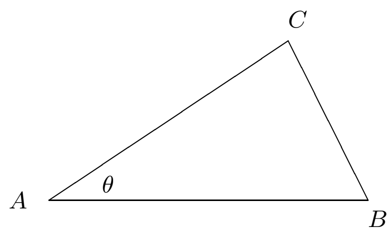
(2) $3\mathbf{i} + 4\mathbf{k}$

(3) $a\mathbf{i} - \mathbf{j} + c\mathbf{k}$

sol. (1) $(1/\sqrt{3})\,\mathbf{i} + (1/\sqrt{3})\,\mathbf{j} + (1/\sqrt{3})\,\mathbf{k}$
(2) $(3/5)\,\mathbf{i} + (4/5)\,\mathbf{k}$
(3) $(a/\sqrt{1+a^2+c^2})\,\mathbf{i} - (1/\sqrt{1+a^2+c^2})\,\mathbf{j} + (c/\sqrt{1+a^2+c^2})\,\mathbf{k}$

Distance $\mathbf{b} - \mathbf{a}$.

[1] $\mathbf{u}/\|\mathbf{u}\|$ is $(1/\|\mathbf{u}\|)\,\mathbf{u}$.



$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB|\,|AC| \cos\theta$$

그림 1.11: law of cosine

angle between two vectors

Definition 2.9. Let $\mathbf{a}, \mathbf{b}$ be two nonzero vectors.

Proposition 2.10. *Let $\mathbf{a}, \mathbf{b}$ be two nonzero vectors and let θ be the angle between them. Then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

Hence

$$\theta = \cos^{-1}\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

Proof. Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AC}$ then $\mathbf{a} - \mathbf{b} = \overrightarrow{CB}$. Since $\angle CAB = \theta$ by law of cosine (fig 1.16) Since

$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

we have

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta = \frac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{b} - \mathbf{a}\|^2)$$

Hence by example 2.4 (4)

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

holds. □

Corollary 2.11. *Two nonzero vector $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.*

Proof. Since $\cos(\pi/2) = 0$ this is trivial by 2.10. □

Example 2.12. Find the angle between $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

sol. By proposition 1.2.10,

$$\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+\mathbf{k}\|} = \frac{-1+2+2}{\sqrt{1+1+4}\sqrt{1+4+1}} = \frac{3}{6} = \frac{1}{2}$$

Hence the angle is $\cos^{-1}(1/2) = \pi/3$.

Corollary 2.13. *Given two points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, the area of the triangle OAB is*

$$\frac{1}{2}\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$$

Proof. Let $\overrightarrow{OA} = \mathbf{u}$, $\overrightarrow{OB} = \mathbf{v}$, $\angle BOA = \theta$. Then the area of $\triangle OAB$ is

$$\begin{aligned}
&\frac{1}{2}|OA|\,|OB|\,\sin\theta \\
=& \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\,\sqrt{1-\cos^2\theta} \\
=& \frac{1}{2}\sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2} \\
=& \frac{1}{2}\sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2} \\
=& \frac{1}{2}\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}
\end{aligned}$$

□

Example 2.14. Find the area of the triangle wit vertices $A(a,0,0), B(0,b,0), C(0,0,c)$.

sol. Shift(translate) A to the origin, then B, C are moved to $(-a, b, 0)$, $(-a, 0, c)$. Hence

$$\frac{1}{2}\sqrt{(bc-0)^2+(0+ac)^2+(0+ab)^2} = \frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$$

Theorem 2.15 (Cauchy-Schwarz inequality)**.** *For any two vectors $\mathbf{u}$, $\mathbf{v}$*

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

holds, and the equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Proof. We may assume $\mathbf{u}$, $\mathbf{v}$ are nonzero. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, $|\cos\theta| \le 1$. Then by prop 2.10

$$|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

holds. Since $\|\mathbf{u}\|\,\|\mathbf{v}\| \ne 0$, if equality holds $|\cos\theta| = 1$ i.e, $\theta = 0$ or π. Hence $\mathbf{u}$ and $\mathbf{v}$ are parallel. □

Remark 2.16. Componentwise, we see

$$(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2)$$

Example 2.17. Show Cauchy-Schwarz inequality for$\mathbf{i}+3\mathbf{j}+2\mathbf{k}$, $-\mathbf{i}+\mathbf{j}$.

sol. Since the inner product and lengths are

$$(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}) = -1+3 = 2,$$
$$\|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\| = \sqrt{1+9+4}\,\sqrt{1+1} = \sqrt{28} = 2\sqrt{7}$$

we have

$$|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j})| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|\,\|-\mathbf{i}+\mathbf{j}\|$$

Theorem 2.18 (Triangle inequality)**.** *For any two vector* $\mathbf{u}$, $\mathbf{v}$ *it holds that*

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

and equality holds when $\mathbf{u}$, $\mathbf{v}$ *are parallel and having same direction.*

Proof.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$$

By C-S

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\|+\|\mathbf{v}\|)^2$$

Equality holds iff

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$$

i.e, the angle is 0. □

Example 2.19. Show triangle inequality for $-\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

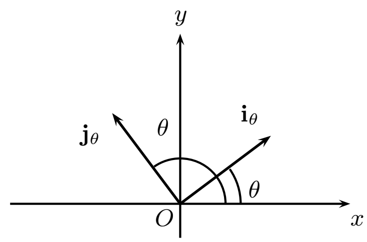


그림 1.12: $\mathbf{i}_\theta$ and $\mathbf{j}_\theta$

sol. Sum and difference is

$$\begin{aligned}\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| &= \|4\mathbf{j}+2\mathbf{k}\| \;=\; \sqrt{16+4} \\ &= 2\sqrt{5} \;=\; 4.4721\ldots \\ \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\| &= \sqrt{1+9+4}+\sqrt{1+1} \\ &= \sqrt{14}+\sqrt{2} \;=\; 5.1558\ldots\end{aligned}$$

Hence

$$\|(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(-\mathbf{i}+\mathbf{j})\| \le \|\mathbf{i}+3\mathbf{j}+2\mathbf{k}\|+\|-\mathbf{i}+\mathbf{j}\|$$

Definition 2.20. If two vectors $\mathbf{u}$, $\mathbf{v}$ satisfy $\mathbf{u}\cdot\mathbf{v}=0$ then we say they are orthogonal **orthogonal**.

Example 2.21. for any real θ two vectors are $\mathbf{i}_\theta = (\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}$, $\mathbf{j}_\theta = -(\sin\theta)\mathbf{i}+(\cos\theta)\mathbf{j}$ orthogonal.

Example 2.22. Find a (unit size) vector orthogonal to $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+9\mathbf{k}$.

sol. Let $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be the desired vector. Then a, b, c are determined by

$$\begin{aligned}2a-b+3c&=0\\ a+2b+9c&=0\\ a^2+b^2+c^2&=1\end{aligned}$$

Hence

$$\pm\frac{1}{\sqrt{19}}(3\mathbf{i}+3\mathbf{j}-\mathbf{k})$$

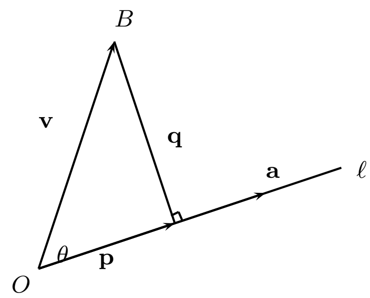


그림 1.13: Projection

Orthogonal projection

If $\mathbf{v}$ is a vector and ℓ is a line through origin in the direction of $\mathbf{a}$, we may define the **orthogonal projection** of $\mathbf{v}$ on $\mathbf{a}$ to be the vector $\mathbf{p}$ given in figure. $\mathbf{p}$ is a scalar multiple of $\mathbf{a}$. There is a c such that $\mathbf{p} = c\mathbf{a}$. Thus,

$$\mathbf{v} = c\mathbf{a} + \mathbf{q}$$

Take inner product with $\mathbf{a}$,

$$\mathbf{a} \cdot \mathbf{v} = c\mathbf{a} \cdot \mathbf{a}$$

so $c = (\mathbf{a} \cdot \mathbf{v})/(\mathbf{a} \cdot \mathbf{a})$. Hence

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2} \cdot \mathbf{a}.$$

The length of $\mathbf{p}$ is

$$\|\mathbf{v}\| \cos\theta.$$

Definition 2.23. For nonzero vector $\mathbf{v}$ and any vector $\mathbf{a}$,we define

$$\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\|^2} \cdot \mathbf{a}.$$

We call it **orthogonal projection of v onto a**.

Example 2.24. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$.

sol. The orthogonal projection is

$$\begin{aligned}\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} &= \frac{3\cdot 1+2\cdot 1+(-1)\cdot 2}{9+4+1}(3\mathbf{i}+2\mathbf{j}-\mathbf{k}) \\ &= \frac{9}{14}\mathbf{i}+\frac{6}{14}\mathbf{j}-\frac{3}{14}\mathbf{k}\end{aligned}$$

Theorem 2.25. *For nonzero* $\mathbf{u}$ *and any* $\mathbf{v}$*, we can write* $\mathbf{v}$ *as the sum of two orthogonal vectors* $\mathbf{a}+\mathbf{b}$*. This decomposition is unique.*

Proof. Denote by $\mathbf{a}$ the projection of $\mathbf{v}$ onto $\mathbf{u}$ and let $\mathbf{b}=\mathbf{v}-\mathbf{a}$. Then

$$\begin{aligned}\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ \mathbf{b} &= \mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\ \mathbf{v} &= \mathbf{a}+\mathbf{b}\end{aligned}$$

Since $\mathbf{a}$ is a multiple of $\mathbf{u}$, and

$$\begin{aligned}\mathbf{u}\cdot\mathbf{b} &= \mathbf{u}\cdot\left(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}\right) \\ &= \mathbf{u}\cdot\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\,\mathbf{u}\cdot\mathbf{u} \\ &= \mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v} \\ &= 0\end{aligned}$$

Hence $\mathbf{b}$ is orthogonal to $\mathbf{u}$. Assume there is real number α s.t. $\mathbf{v}=\alpha\mathbf{u}+\mathbf{c}$, $\mathbf{u}\cdot\mathbf{c}=0$ hold. Then

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot(\alpha\mathbf{u}+\mathbf{c})=\alpha\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{c}=\alpha\|\mathbf{u}\|^2$$

we see

$$\begin{aligned}\alpha\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \;=\; \mathbf{a} \\ \mathbf{c} &= \mathbf{v}-\alpha\mathbf{u} \;=\; \mathbf{v}-\mathbf{a} \;=\; \mathbf{b}\end{aligned}$$

The decomposition of $\mathbf{v}$ in $\mathbf{u}$ and orthogonal to $\mathbf{u}$ is unique. □

Definition 2.26. The vector $\mathbf{a}$ is called a **component parallel to u** and $\mathbf{b}$ is **component orthogonal to u**.(**orthogonal complement**)

Example 2.27. Express $\mathbf{v} = 3\mathbf{i}+5\mathbf{j}+\mathbf{k}$ as parallel and orthogonal components to $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

sol. Let $\mathbf{a}$ be the projection of $\mathbf{v}$ onto $\mathbf{u}$ and $\mathbf{b} = \mathbf{v} - \mathbf{a}$. Then

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{1\cdot 3 + 2\cdot 5 + (-1)\cdot 1}{1+4+1}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\
&= 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \\
\mathbf{b} &= (3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\
&= \mathbf{i} + \mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Here $\mathbf{a}$ is parallel to $\mathbf{u}$, $\mathbf{b}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v} = \mathbf{a} + \mathbf{b}$. ∎

Triangle inequality

Theorem 2.28. *For* $\mathbf{a}, \mathbf{b}$*, we have*

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Use C-S.

Physical applications

Displacement : If an object has moved from P to Q, then $\vec{PQ}$ is the displacement.

Example 2.29. A ship is running on the sea at the speed of $20km$ to north. but the current is flowing at the speed of $20km$ to the east, then in one hr, the displacement of the ship is $(20\sqrt{2}, 20\sqrt{2})$.

제 3 절 Matrices and Cross product

2×2 matrix

The array of numbers $a_{11},\ a_{12},\ a_{21},\ a_{22}$ in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is called 2×2 **matrix** and

$$[a_{11}\ a_{12}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

are the first row and second column. The real number $a_{11}a_{22} - a_{12}a_{21}$ is **determinant** and denoted by

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 3.1. Find determinant of 2×2 matrices.

$$\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5, \quad \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = 0 - (-3) = 3, \quad \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 - (-4) = 5$$

Proposition 3.2. *The area of parallelogram having two vectors* $a\mathbf{i} + b\mathbf{j}$ *and* $c\mathbf{i} + d\mathbf{j}$ *as two sides is the absolute value* $|ad - bc|$ *of*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Proof. Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$ and θ be the angle between

$$\begin{aligned} \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1 - \cos^2\theta} \\ &= \sqrt{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2} \\ &= \sqrt{a^2d^2 + b^2c^2 - 2abcd} \\ &= |ad - bc| \end{aligned}$$

□

3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here

$$[a_{31}\ a_{32}\ a_{33}], \qquad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

are third row and second column. The **determinant** is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{1.1}$$

Example 3.3. 3×3

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 1 \begin{vmatrix} 4 & 8 \\ 9 & 27 \end{vmatrix} - 1 \begin{vmatrix} 2 & 8 \\ 3 & 27 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 36 - 30 + 6 = 12$$

Definition 3.4. If we exchange rows and columns of the following matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

to get

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

then resulting matrices are called **transpose**.

Properties of determinant

Theorem 3.5. *(1) Determinant of transposed matrix is the same the Determinant of original matrix.*

(2) If we exchange ant two row(column), then determinant changes signs.

(3) $|det(\alpha A)| = \alpha^n |det(A)|$

(4) Adding a scalar multiple of row (column) to another row (column) does not change determinant.

Proof. (1) For 2×2

$$\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

For 3×3

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) \\
&\qquad + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} \\
&\qquad + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\
&\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

(4) 2×2 case is easy.

For 3×3, we see by expanding w.r.t. first row

$$\begin{vmatrix} a_{11} + ta_{21} & a_{12} + ta_{22} & a_{13} + ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11} + ta_{21})\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12} + ta_{22})\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (a_{13} + ta_{23}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$+ t \left(a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \right)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + t \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Exchange second and third rows, do not change the value. By (2) there must be a sign change. Hence it is 0.

$$\begin{vmatrix} a_{11} + ta_{21} & a_{12} + ta_{22} & a_{13} + ta_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Hence (4) holds. □

The RHS of 1.1 is expansion w.r.t **first row.** By theorem 3.5, (1), (2), we can expand w.r.t. any row or column, except we multiply $(-1)^{i+j}$. So if we expand w.r.t 2nd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

So if we expand w.r.t 3rd row

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Corollary 3.6. *(1) Determinant of a matrix whose whole row is zero is zero. (2) If any two row (columns) are equal, the determinant is zero.*

Example 3.7. The followings are expanded w.r.t 2nd, 3rd row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 60 + 48 = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} = 0\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} = 0 + 48 + 0 = 48$$

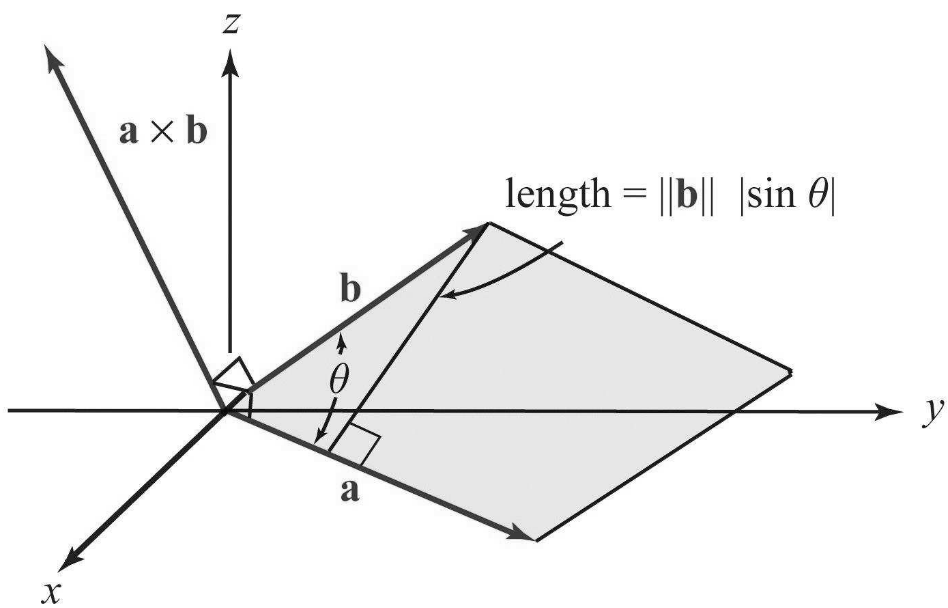
Cross product

Definition 3.8. For $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Using definition of determinant (1.1) symbolically, we have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Example 3.9. $\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}.$

Example 3.10. Find $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

sol. By definition of Cross product

$$(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

Triple product

To give a geometric interpretation of cross product, we introduce triple product of three vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

as $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Then we see

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}\right)\cdot(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c_3 \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\end{aligned}$$

If $\mathbf{c}$ is a vector in the plane spanned by $\mathbf{a},\mathbf{b}$ then the third row in the determinant is a linear combination of the first and second row, and hence $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$. In other words, *the vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to any vector in the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.*

We compute length of $\mathbf{a}\times\mathbf{b}$.

$$\begin{aligned}\|\mathbf{a}\times\mathbf{b}\|^2 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \\ &= (a_2b_3-a_2b_2)^2+(a_1b_3-b_1a_3)^2+(a_1b_2-b_1a_2)^2 \\ &= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2\end{aligned}$$

Hence

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta.$$

So we conclude that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane $\mathcal{P}$ spanned by $\mathbf{a}$ and $\mathbf{b}$ with length $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$.(With right hand rule) Hence we obtain

Theorem 3.11. *$\|\mathbf{a}\times\mathbf{b}\|$ is the area of parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.*

Theorem 3.12 (Cross Product)**.** *For $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and scalar α, β, it holds that*

(1) $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, *the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.*

(2) $\mathbf{a}\times\mathbf{b}$ *is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and the triple $(\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b})$ form a right-handed rule.*

Component formula:

$$(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Algebraic rules:

(1) $\mathbf{a}\times\mathbf{b} = 0$, *iff* $\mathbf{a}$ $\mathbf{b}$ *are parallel or zero.*

(2) $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$

(3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

(4) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

(5) $(\alpha\mathbf{a})\times\mathbf{b} = \alpha(\mathbf{a}\times\mathbf{b})$.

Multiplication rules:

(1) $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$.

(2) $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

(3) $\mathbf{i}\times\beta\mathbf{i} = \mathbf{j}\times\beta\mathbf{j} = \mathbf{k}\times\beta\mathbf{k} = 0$

Example 3.13. Find $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}-2\mathbf{k})$.

sol. $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}-2\mathbf{k}) = \mathbf{i}\times j - 2\mathbf{i}\times k + \mathbf{j}\times j - 2\mathbf{j}\times k = -2\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

Theorem 3.14 (Cross product II)**.**

(1) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$. *In particular,* $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

(2) If is θ *the angle between* $\mathbf{u}$ *and* $\mathbf{v}$, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$. *Hence nec. and suff. condition for* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel is* $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

(3) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

(4) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$, *i.e* $\mathbf{u}\times\mathbf{v}$ *is orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. Let $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(1) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ was shown before.

So $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(2) Since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$, we have by (1)

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\| &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sqrt{1 - \cos^2\theta} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta \end{aligned}$$

(3)

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \right) \cdot (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right) \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \end{aligned}$$

(4) Using (3) and corollary 3.6, we see

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

□

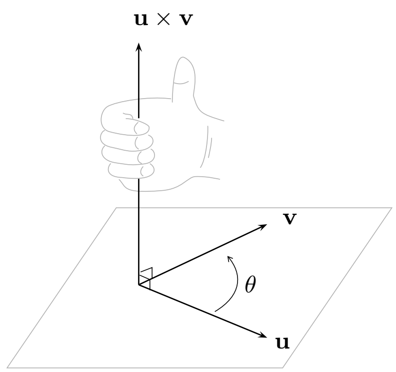


그림 1.14: right handed rule

Geometric Meaning of Cross Product

Geometry of Determinant

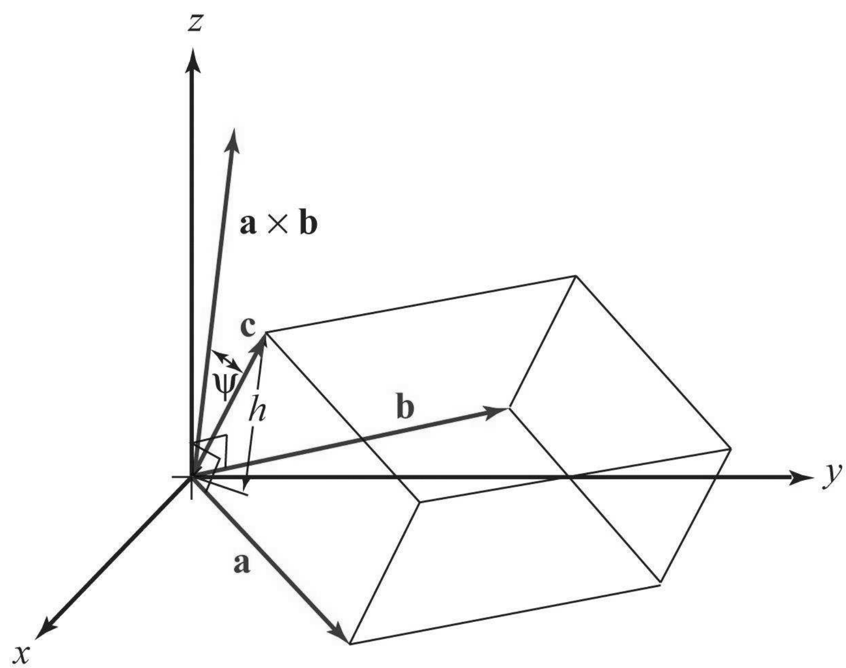
2×2 matrix: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$\|\mathbf{a} \times \mathbf{b}\|$ is the absolute value of the determinant and is the area of parallelogram determined by two vectors.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

Example 3.15. Find the area of triangle with vertices at $(1,1), (0,2)$ and $(3,2)$. Sol. Two sides are $(0,2) - (1,1) = (-1,1)$ and $(3,2) - (1,1) = (2,1)$. Thus $\frac{1}{2} \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{3}{2}$.



Proposition 3.16. *The volume of parallelepiped with sides* **a**, **b**, **c** *is give by the absolute value of triple product* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *which is the determinant*

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof. Consider a parallelogram with two sides **a**, **b** as bottom of the parallelepiped. Then the height is length of the orthogonal projection of **c** onto $\mathbf{a} \times \mathbf{b}$ which is $\left\| \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}{\|\mathbf{a}\times\mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\|$. Hence the volume is

$$Area(bottom) \times height = \|\mathbf{a} \times \mathbf{b}\| \left\| \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}{\|\mathbf{a} \times \mathbf{b}\|^2} \mathbf{a} \times \mathbf{b} \right\| = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$

This is nothing but the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

□

Example 3.17. Three points $A(1,2,3)$, $B(0,1,2)$, $C(0,3,2)$ are given. Find the volume of hexahedron having three vectors OA, OB, OC as sides.

sol. By proposition 3.16, we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -4$$

Equations of Planes

Let $\mathcal{P}$ be a plane and $P_0 = (x_0, y_0, z_0)$ a point on that plane, and suppose that $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector. Let $P = (x, y, z)$ be any point in $\mathbb{R}^3$. Then P lies in the plane iff the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to $\mathbf{n}$, that is, $\overrightarrow{P_0P} \cdot \mathbf{n} = 0$. In other words,

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0.$$

Proposition 3.18. *Equation of plane through (x_0, y_0, z_0) that has normal vector $\mathbf{n}$ is*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz - D = 0$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

Example 3.19. Find the equation of plane through the points $A(-3, 0, -1)$, $B(-2, 3, 2)$, $C(1, 1, 3)$.

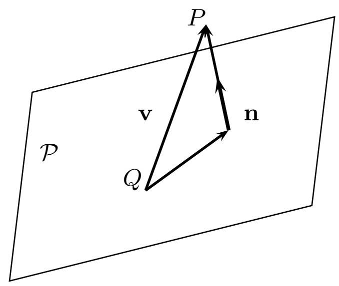


그림 1.15: Distance from a point to plane

sol. Draw some graph describing the normal vector.
Find a vector **n** orthogonal to plane.

$$\begin{aligned}
\mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-(-3) & 3-0 & 2-(-1) \\ 1-(-3) & 1-0 & 3-(-1) \end{vmatrix} \\
&= \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \mathbf{k} \\
&= 9\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}
\end{aligned}$$

By proposition 3.18, the equation is

$$9(x+3) + 8(y-0) - 11(z+1) = 0$$

or $9x + 8y - 11z + 16 = 0$.

Distance from a point to plane

Proposition 3.20. *The distance from $P(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is*

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Proof. If $Q(x_0, y_0, z_0)$ lies in the plane, the distance from P to the plane is the orthogonal projection of $\overrightarrow{PQ}$ along **n**. Note that from $A(x - x_0) + B(y - y_0) +$

$C(z - z_0) = 0$, we see $\mathbf{n} // A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Normalizing,

$$\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}.$$

Hence length of the orthogonal projection of $\overrightarrow{PQ}$ along $\mathbf{n}$ is

$$\begin{aligned}
\left\| \frac{\mathbf{n} \cdot \overrightarrow{PQ}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| &= \frac{|\mathbf{n} \cdot \overrightarrow{PQ}|}{\|\mathbf{n}\|} \\
&= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{|Ax_0 + By_0 + Cz_0 - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}} \\
&= \frac{|-D - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}
\end{aligned}$$

□

Example 3.21. Find the distance from $(3, 4, -2)$ to the plane $2x - y + z - 4 = 0$.

sol. Using above proposition, distance is

$$\frac{|2 \cdot 3 - 1 \cdot 4 + 1 \cdot (-2) - 4|}{\sqrt{4 + 1 + 1}} = \frac{|-4|}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

Example 3.22. Find a vector perpendicular to the plane $4x - 3y + z - 4 = 0$ and express it as a cross product of two unit orthogonal vectors.

sol. Let $\mathcal{S}$ the given plane. By proposition 3.18 we see $4\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is orthogonal to $\mathcal{S}$. Hence

$$\mathbf{n} = \pm\frac{4\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{4^2+(-3)^2+1^2}} = \pm\frac{1}{\sqrt{26}}(4\mathbf{i}-3\mathbf{j}+\mathbf{k})$$

Now in order to express this as a cross product of two vectors, we need to choose three points $(1,0,0), (0,0,4), (2,1,-1)$ in $\mathcal{S}$. Then we obtain two vectors

$$\begin{aligned}\mathbf{u} &= (1,0,0)-(2,1,-1) = -\mathbf{i}-\mathbf{j}+\mathbf{k}\\ \mathbf{v} &= (0,0,4)-(2,1,-1) = -2\mathbf{i}-\mathbf{j}+5\mathbf{k}\end{aligned}$$

which are parallel to the plane $\mathcal{S}$. Let us orthogonalize them. Let $\mathbf{a}$, $\mathbf{b}$ be the parallel/orthogonal component of $\mathbf{v}$ to $\mathbf{u}$, i.e, Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$. Then find $\mathbf{b}$ as $\mathbf{v}-\mathbf{a}$.

$$\begin{aligned}\mathbf{a} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{8}{3}(-\mathbf{i}-\mathbf{j}+\mathbf{k})\\ \mathbf{b} &= \mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = (-2\mathbf{i}-\mathbf{j}+5\mathbf{k})-\frac{8}{3}(-\mathbf{i}-\mathbf{j}+\mathbf{k})\\ &= \frac{1}{3}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})\end{aligned}$$

Now normalize them.

$$\mathbf{a}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}+\mathbf{k}) \quad \mathbf{b}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{1}{\sqrt{78}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})$$

We can check that

$$\begin{aligned}\mathbf{a}_1\cdot\mathbf{b}_1 &= \frac{(-1)\cdot 2+(-1)\cdot 5+1\cdot 7}{\sqrt{3}\cdot\sqrt{78}} = 0\\ \mathbf{a}_1\times\mathbf{b}_1 &= \frac{1}{3\sqrt{26}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 1\\ 2 & 5 & 7\end{vmatrix}\\ &= \frac{1}{3\sqrt{26}}\left(\begin{vmatrix}-1 & 1\\ 5 & 7\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ 2 & 7\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ 2 & 5\end{vmatrix}\mathbf{k}\right)\\ &= -\frac{1}{\sqrt{26}}(4\mathbf{i}-3\mathbf{j}+\mathbf{k})\end{aligned}$$

Also it can be written as

$$\frac{-\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{3}}\times\frac{2\mathbf{i}+5\mathbf{j}+7\mathbf{k}}{\sqrt{78}} = -\frac{4\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{26}}$$

□

제 4 절 Cylindrical and spherical coordinate

Cylindrical coordinate system

Given a point $P = (x, y, z)$, we can use polar coordinate for (x, y)-plane. Then it holds that

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

we say (r, θ, z) is **cylindrical coordinate** of P. The set of all points $r = a$ is

$$\{(x, y, z) \mid x^2 + y^2 = a^2\}$$

This is a cylinder (Fig 1.16).

r and θ satisfies

$$r^2 = x^2 + y^2, \quad \frac{y}{x} = \tan\theta$$

This expression is not unique. However, Any nonzero point is uniquely determined in the range of $0 \le \theta < 2\pi$, $r > 0$ as follows:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \tan^{-1}(y/x) & (x > 0,\ y \ge 0) \\ \pi + \tan^{-1}(y/x) & (x < 0) \\ 2\pi + \tan^{-1}(y/x) & (x > 0,\ y < 0) \\ \pi/2 & (x = 0,\ y > 0) \\ 3\pi/2 & (x = 0,\ y < 0) \end{cases} \tag{1.2}$$

Here $-\pi/2 < \tan^{-1}(y/x) < \pi/2$.

Example 4.1. Change cylindrical coordinate $(6, \pi/3, 4)$ to Cartesian coordinate.

sol.

$$x = 6\cos(\pi/3) = 3, \quad y = 6\sin(\pi/3) = 3\sqrt{3}, \quad z = 4$$

$(x, y, z) = (3, 3\sqrt{3}, 4)$.

□

Example 4.2. Change $(4, -4, 3)$ to cylindrical coordinate.

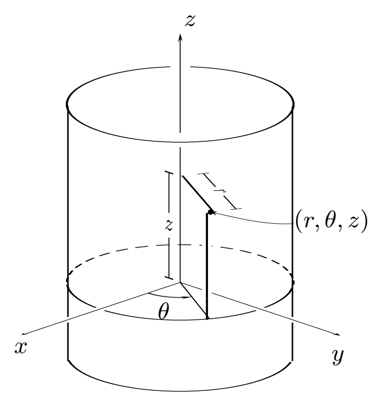


그림 1.16: cylindrical coordinate

sol.

$$r = \sqrt{16+16} = 4\sqrt{2}, \quad \theta = 2\pi + \tan^{-1}(-1) = 2\pi - \pi/4 = 7\pi/4$$

$(r, \theta, z) = (4\sqrt{2}, 7\pi/4, 3).$

Example 4.3. Change the equation $x^2+y^2-z^2 = 1$ to cylindrical coordinate..

sol. $r^2 - z^2 = 1.$

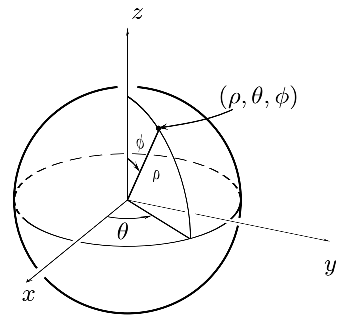
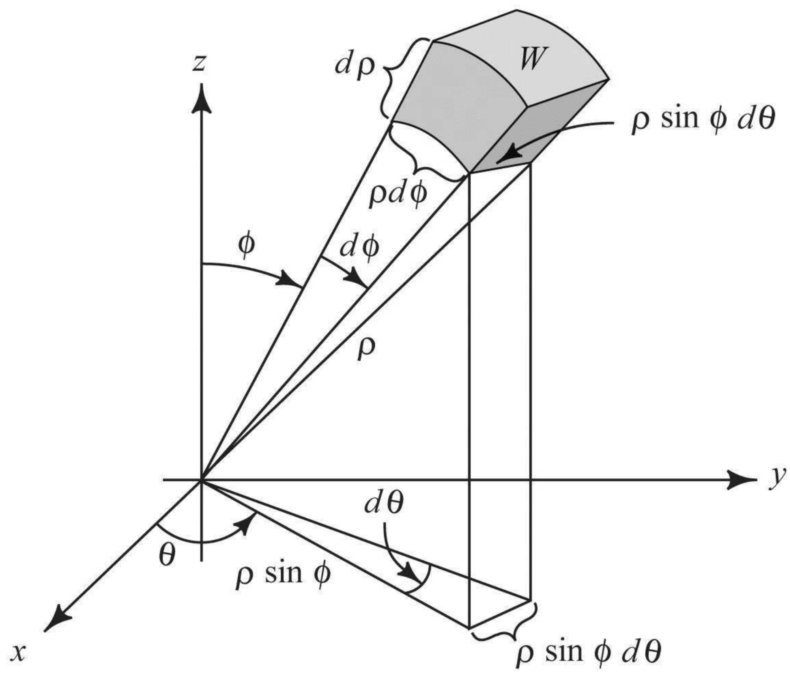


그림 1.17: Spherical-coordinate



Spherical coordinate system

For $P = (x, y, z)$ we have

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{pmatrix} \rho \geq 0 \\ 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{pmatrix}$$

we call (ρ, θ, ϕ) to be the **spherical coordinate** of P. Writing ρ, θ, ϕ in x,

y, z and let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then we have

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \begin{cases} \tan^{-1}(y/x) & (x > 0,\ y \geq 0) \\ \pi + \tan^{-1}(y/x) & (x < 0) \\ 2\pi + \tan^{-1}(y/x) & (x > 0,\ y < 0) \\ \pi/2 & (x = 0,\ y > 0) \\ 3\pi/2 & (x = 0,\ y < 0) \end{cases}$$
$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|}\right)$$

Example 4.4. (1) Find spherical coord. of $(1, -1, 1)$ and plot.

(2) Find cartesian coord. of $(3, \pi/6, \pi/4)$.

(3) Find spherical coord. of $(2, -3, 6)$.

(4) Find spherical coord. of $(3, -3, \sqrt{6})$.

sol. (1) $\rho = \sqrt{3}$.

$$\theta = \arctan(\frac{y}{x}) = \frac{7\pi}{4}$$
$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{1}{\sqrt{3}}) \approx 0.955 \approx 54.74^o$$

(2) $\rho = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{2^2 + (-3)^2 + 6^2} = 7.$

$$\theta = 2\pi + \tan^{-1}(-3/2)$$
$$\phi \cos^{-1}\frac{6}{7}$$

(3) $\rho = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{2^2 + (-3)^2 + 6^2} = 7.$ (4)

$$\rho = \sqrt{9 + 9 + 6} = 2\sqrt{6}$$
$$\theta = 2\pi + \tan^{-1}(-1) = 2\pi - (-\frac{\pi}{4}) = \frac{7\pi}{4}$$
$$\phi = \cos^{-1}\frac{\sqrt{6}}{2\sqrt{6}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

Hence cylindrical coordinate is $(2\sqrt{6}, 7\pi/4, \pi/3)$.

$\lrcorner$

Example 4.5. Express the surface (1) $xz = 1$ and (2) $x^2 + y^2 - z^2 = 1$ in spherical coordinate.

sol. (1) Since $xz = \rho^2 \sin\phi \cos\theta \cos\phi = 1$, we have the equation

$$\rho^2 \sin 2\phi \cos\phi = 2.$$

(2) Since $x^2 + y^2 - z^2 = x^2 + y^2 + z^2 - 2z^2 = \rho^2 - 2(\rho\cos\phi)^2 = \rho^2(1 - 2\cos^2\phi)$, the equation is $1 + \rho^2 \cos 2\phi = 0$.

제 5 절 n-dim Euclidean space

Vectors in n -dim space

The set of all points with n-coordinates

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \text{ real}\}$$

is called n**-dimensional Euclidean space**. Addition and scalar multiplication can be defined as

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$s(a_1, a_2, \ldots, a_n) = (sa_1, sa_2, \ldots, sa_n)$$

The identity $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$ is **zero element**. The inverse of $(a_1, a_2, \ldots, a_n)$ is $(-a_1, -a_2, \ldots, -a_n)$, or $-(a_1, a_2, \ldots, a_n)$.

For two points $P(a_1, a_2, \ldots, a_n)$ and $Q(b_1, b_2, \ldots, b_n)$, the set

$$\overline{PQ} = \{(1-t)(a_1, a_2, \ldots, a_n) + t(b_1, b_2, \ldots, b_n) \mid 0 \le t \le 1\}$$

is called the **line segment** PQ and

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

is **length** PQ. Also the set

$$\overleftrightarrow{PQ} = \{(1-t)(a_1, a_2, \ldots, a_n) + t(b_1, b_2, \ldots, b_n) \mid -\infty < t < \infty\}$$

is line PQ.

For three points $P(a_1, \dots, a_n)$ $Q(b_1, \dots, b_n)$, $R(c_1, \dots, c_n)$ not on the same line, the set

$$\{r(a_1, \dots, a_n) + s(b_1, \dots, b_n) + t(c_1, \dots, c_n) \mid -\infty < r, s, t < \infty, r + s + t = 1\}$$

is called the plane **determined by** $P,\ Q,\ R$.

Basis vector

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \dots, 0)\\
\mathbf{e}_2 &= (0, 1, 0, \dots, 0)\\
\mathbf{e}_3 &= (0, 0, 1, \dots, 0)\\
&\vdots\\
\mathbf{e}_n &= (0, 0, 0, \dots, 1)
\end{aligned}$$

Any vector can be written as a scalar combination of $\mathbf{e}_1,\ \mathbf{e}_2, \dots,\ \mathbf{e}_n$.

$$(a_1, a_2, \dots, a_n) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$$

The vectors $\mathbf{e}_1,\ \mathbf{e}_2, \dots,\ \mathbf{e}_n$ are **standard basis vector** of $\mathbb{R}^n$. [2] in $\mathbb{R}^3$

Theorem 5.1. **(i)** $(\alpha\mathbf{x} + \beta\mathbf{y}) \cdot \mathbf{z} = \alpha\mathbf{x} \cdot \mathbf{z} + \beta\mathbf{y} \cdot \mathbf{z}$ *(associate law)*

(ii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ *(commutative law)*

(iii) $\mathbf{x} \cdot \mathbf{x} \geq 0$

(iv) $\mathbf{x} \cdot \mathbf{x} = 0$ *iff* $\mathbf{x} = 0$

Example 5.2. Let $\mathbf{u} = 3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4$, $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4$ be in $\mathbb{R}^4$. Express $2\mathbf{u} - 7\mathbf{v}$ using standard basis vector.

[2] By definition 1.7 $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$

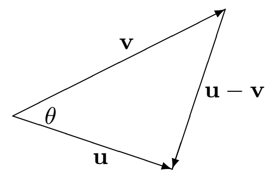


그림 1.18: angle between two vectors

sol. Using standard basis vector, $2\mathbf{u} - 7\mathbf{v}$ is

$$\begin{aligned} 2\mathbf{u} - 7\mathbf{v} &= 2(3\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_4) - 7(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4) \\ &= (6\mathbf{e}_1 - 8\mathbf{e}_2 + 4\mathbf{e}_4) + (-7\mathbf{e}_1 - 14\mathbf{e}_2 - 14\mathbf{e}_3 + 21\mathbf{e}_4) \\ &= (6-7)\mathbf{e}_1 + (-8-14)\mathbf{e}_2 + (0-14)\mathbf{e}_3 + (4+21)\mathbf{e}_4 \\ &= -\mathbf{e}_1 - 22\mathbf{e}_2 - 14\mathbf{e}_3 + 25\mathbf{e}_4 \end{aligned}$$

For two vector $\mathbf{u} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$, $\mathbf{v} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n$, **inner product** is defined as

$$\mathbf{u} \cdot \mathbf{v} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

This satisfies proposition 2.3. The size of $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = (a_1^2 + \cdots + a_n^2)^{1/2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Law of Cosine

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$

The angle between $\mathbf{u}$ and $\mathbf{v}$ is (Fig 1.23).

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \cos^{-1} \frac{a_1b_1 + \cdots + a_nb_n}{(a_1^2 + \cdots + a_n^2)^{1/2}(b_1^2 + \cdots + b_n^2)^{1/2}}$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

Example 5.3. Find the inner product of $\mathbf{u} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$, $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2 - 3\mathbf{e}_3 - \mathbf{e}_4$.

sol.

$$\mathbf{u} \cdot \mathbf{v} = 2 - 2 - 9 - 2 = -11$$

Example 5.4. Find the angle between $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4$, $\mathbf{v} = -\mathbf{e}_2 - \mathbf{e}_3 + 2\mathbf{e}_4$.

sol. The angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos^{-1} \frac{0+1+0+2}{\sqrt{(1+1+0+1)(0+1+1+4)}} = \cos^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

Theorem 5.5 (Cauchy-Schwarz inequality)**.** *For any two vectors* $\mathbf{u}$, $\mathbf{v}$ *in* n-*dim space the following holds. Equality holds iff* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel.*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Theorem 5.6 (Triangle inequality)**.** *For any two vectors* $\mathbf{u}$, $\mathbf{v}$ *in* n-*dim space the following holds. Equality holds iff* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel and same direction.*

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

General matrix

Let m, n be any natural numbers. The arrays a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called $m \times n$ **matrix** and denote by

$$\begin{bmatrix} a_{ij} \end{bmatrix}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}, \quad \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} \text{ or } [a_{ij}]$$

If $m = 1$, then $1 \times n$ matrix consists of one row and is called **row vector**, and if $n = 1$ then $m \times 1$ matrix is **column vector**. If $m = n$, it is called **square matrix**. a_{ij} is called ij**-entry** . The $1 \times n$ matrix

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

is i**-th row vector**, $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

is j**-th column vector**.

Example 5.7. What is 4-th row and second column of the 4×3 matrix?

$$\begin{bmatrix} 0 & -2 & 12 \\ 3 & 1 & 4 \\ -1 & 0 & 5 \\ 1 & -3 & 7 \end{bmatrix}$$

sol. 4-th row and second column is

$$\begin{bmatrix} 1 & -3 & 7 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$$

Product(multiplication)

$A = [a_{ij}]$ is $m \times n$ matrix and $B = [b_{kl}]$ is $n \times p$ matrix. Then the $m \times p$ matrix

$$\left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{\substack{1 \le i \le m \\ 1 \le j \le p}}$$

is the **product** of A and B denoted by AB. In other words, the product of A and B is AB and its ij-component is the inner product of $i-th$ row of A and $j-th$ column of B.

Example 5.8. Product of 2×3 and 3×4 matrices

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & -2 \\ -2 & 1 & 5 & -3 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 19 & -12 \\ 6 & -1 & -6 & 3 \end{bmatrix}$$

Example 5.9. Product of 1×3 and 3×2 matrices

$$\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 1 & -3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 13 \end{bmatrix}$$

Example 5.10. Product of 3×4 and 4×1 matrices

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ -1 & 2 & 0 & -3 \\ 2 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y + 3z + w \\ -x + 2y - 3w \\ 2x + z + 4w \end{bmatrix}$$

Definition 5.11. The following $n \times n$ matrix is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$n \times n$ **identity matrix** and denote it by I_n.

Lemma 5.12. *For nay $m \times n$ matrix A and $n \times p$ matrix B, we have*

$$A\,I_n = A, \qquad I_n\,B = B$$

Also, for any $n \times n$ matrix A, it holds that

$$A\,I_n = I_n\,A = A$$

I_n is identity element in multiplication.

Example 5.13.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 5.14. If for any vector $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is defined by

(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(2) $T(\alpha\,\mathbf{u}) = \alpha\,T(\mathbf{u})$

we say T **linear transformation.**

Example 5.15. Express linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ using standard basis vector.

Since any vector in $\mathbb{R}^n$ can be written as $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ and T is determined by the values at these vectors.

$$\begin{aligned} T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) &= T(a_1\mathbf{e}_1) + T(a_2\mathbf{e}_2) + \cdots + T(a_n\mathbf{e}_n) \\ &= a_1T(\mathbf{e}_1) + a_2T(\mathbf{e}_2) + \cdots + a_nT(\mathbf{e}_n) \end{aligned}$$

Since $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$ are in $\mathbb{R}^m$, we can write it as linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$. Hence there are t_{ij} $(1 \le i \le m, 1 \le j \le n)$ s.t.

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} t_{ij}\mathbf{e}_i \quad (1 \le j \le n) \tag{1.3}$$

Hence

$$T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n) = \sum_{j=1}^{n} a_jT(\mathbf{e}_j) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} t_{ij}a_j\right)\mathbf{e}_i \tag{1.4}$$

This procedure can be written in matrix form Eq. (1.3). The matrix having t_{ij} as ij-th component

$$\operatorname{mat}(T) = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix}$$

is called **matrix of** T. Let us multiply the column vector $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ to the right of this matrix.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} t_{11}a_1 + t_{12}a_2 + \cdots + t_{1n}a_n \\ t_{21}a_1 + t_{22}a_2 + \cdots + t_{2n}a_n \\ \vdots \\ t_{m1}a_1 + t_{m2}a_2 + \cdots + t_{mn}a_n \end{bmatrix}$$

Compare this with equation (1.4). Then rhs vector has $T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots +$

$a_n\mathbf{e}_n$) as its component. Conversely, any $m \times n$ matrix $[t_{ij}]$ is given, then it determines linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ as in equation (1.4). Hence linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has one-one correspondence with $m \times n$ matrix as follows:

$$\mathrm{mat}\colon T \mapsto \Big[\ \mathbf{e}_i \cdot T(\mathbf{e}_j)\ \Big]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Proposition 5.16. *For two linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ it holds that*

$$\mathrm{mat}(T \circ U) = \mathrm{mat}(T)\ \mathrm{mat}(U)$$

Example 5.17. For the given two linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $U\colon \mathbb{R}^p \to \mathbb{R}^n$ check Proposition 5.16 holds.

$$\begin{aligned} T(x,y,z) &= (3y - z, x + y) \\ U(s,t) &= (2s - t, s + 2t, -3s) \end{aligned}$$

sol. The matrices for T and U are

$$\mathrm{mat}(T) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathrm{mat}(U) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}$$

Hence

$$\mathrm{mat}(T)\ \mathrm{mat}(U) = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 1 \end{bmatrix}$$

On the other hand

$$\begin{aligned} (T \circ U)(s,t) &= T(2s - t, s + 2t, -3s) \\ &= (3(s + 2t) - (-3s), (2s - t) + (s + 2t)) \\ &= (6s + 6t, 3s + t) \end{aligned}$$

So

$$\mathrm{mat}(T \circ U) = \begin{bmatrix} 6 & 6 \\ 3 & 1 \end{bmatrix}$$

Hence the following holds.

$$\mathrm{mat}(T \circ U) = \mathrm{mat}(T)\ \mathrm{mat}(U)$$

Determinant

We have seen 3×3 2×2. Using these, we define determinant of $n \times n$ matrix by induction. We expand w.r.t 1st column.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} +$$

$$\cdots + (-1)^{1+i} a_{1i} \begin{vmatrix} a_{21} & \cdots & a_{2(i-1)} & a_{2(i+1)} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3(i-1)} & a_{3(i+1)} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(i-1)} & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$+(-1)^{1+n} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} \\ \cdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} \end{vmatrix}$$

The i-th term on the right is $(-1)^{1+i} a_{1i}$ times the determinant of $(n-1) \times (n-1)$ obtained by deleting first row and i-column.

Thm 3.5 and corollary 3.6 hold for any square matrices.

Expansion with respect to any row

LetA_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting i-row and j-th column. Expand w.r.t i-th row, we see

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

if we expand w.r.t j-th row, we see

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

Example 5.18. Expand w.r.t 2nd row

$$\begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 4 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & -3 & 0 \end{bmatrix} = -0 \begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 2 \\ 2 & -3 & 0 \end{bmatrix}$$

$$- 0 \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} + 3 \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

$$+ 4 \cdot 2 \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix} + 4(-3) \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$= -2 \cdot 6 + 3 \cdot (-4) + 8 \cdot (-3) - 12 \cdot 5$$

$$= -108$$

Example 5.19. Solve

$$\begin{array}{ccccccc} 3x & + & 2y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ x & + & y & & & = & 3 \end{array}$$

sol. Use Cramer's rule

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

Then $x_1.x_2$ and x_3 are

$$x_1 = \frac{1}{-2} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \quad x_2 = \frac{1}{-2} \begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 4 \quad x_3 = \frac{1}{-2} \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = -4$$

∎