MATHEMATICS-I

VECTOR CALCULUS

I YEAR B.Tech

By

Y. Prabhaker Reddy
Asst. Professor of Mathematics
Guru Nanak Engineering College
Ibrahimpatnam, Hyderabad.

SYLLABUS OF MATHEMATICS-I (AS PER JNTU HYD)

Name of the Unit	Name of the Topic
Unit-I Sequences and Series	1.1 Basic definition of sequences and series 1.2 Convergence and divergence. 1.3 Ratio test 1.4 Comparison test 1.5 Integral test 1.6 Cauchy's root test 1.7 Raabe's test 1.8 Absolute and conditional convergence
Unit-II Functions of single variable	2.1 Rolle's theorem 2.2 Lagrange's Mean value theorem 2.3 Cauchy's Mean value theorem 2.4 Generalized mean value theorems 2.5 Functions of several variables 2.6 Functional dependence, Jacobian 2.7 Maxima and minima of function of two variables
Unit-III Application of single variables	3.1 Radius , centre and Circle of curvature 3.2 Evolutes and Envelopes 3.3 Curve Tracing-Cartesian Co-ordinates 3.4 Curve Tracing-Polar Co-ordinates 3.5 Curve Tracing-Parametric Curves
Unit-IV Integration and its applications	4.1 Riemann Sum 4.3 Integral representation for lengths 4.4 Integral representation for Areas 4.5 Integral representation for Volumes 4.6 Surface areas in Cartesian and Polar co-ordinates 4.7 Multiple integrals-double and triple 4.8 Change of order of integration 4.9 Change of variable
Unit-V Differential equations of first order and their applications	5.1 Overview of differential equations 5.2 Exact and non exact differential equations 5.3 Linear differential equations 5.4 Bernoulli D.E 5.5 Newton's Law of cooling 5.6 Law of Natural growth and decay 5.7 Orthogonal trajectories and applications
Unit-VI Higher order Linear D.E and their applications	6.1 Linear D.E of second and higher order with constant coefficients 6.2 R.H.S term of the form *exp(ax)* 6.3 R.H.S term of the form *sin ax and cos ax* 6.4 R.H.S term of the form exp(ax) v(x) 6.5 R.H.S term of the form exp(ax) v(x) 6.6 Method of variation of parameters 6.7 Applications on bending of beams, Electrical circuits and simple harmonic motion
Unit-VII Laplace Transformations	7.1 LT of standard functions 7.2 Inverse LT –first shifting property 7.3 Transformations of derivatives and integrals 7.4 Unit step function, Second shifting theorem 7.5 Convolution theorem-periodic function 7.6 Differentiation and integration of transforms 7.7 Application of laplace transforms to ODE
Unit-VIII Vector Calculus	8.1 Gradient, Divergence, curl 8.2 Laplacian and second order operators 8.3 Line, surface , volume integrals 8.4 Green's Theorem and applications 8.5 Gauss Divergence Theorem and applications 8.6 Stoke's Theorem and applications

CONTENTS

UNIT-8

VECTOR CALCULUS

- Gradient, Divergence, Curl
- Laplacian and Second order operators
- Line, surface and Volume integrals
- Green's Theorem and applications
- Gauss Divergence Theorem and application
- Stoke's Theorem and applications

Differentiation of Vectors

Scalar: A Physical Quantity which has magnitude only is called as a Scalar.
Ex: Every Real number is a scalar.

Vector: A Physical Quantity which has both magnitude and direction is called as Vector.
Ex: Velocity, Acceleration.

Vector Point Function: Let "D" be a Domain of a function, then if for each variable $t \in D, \exists$ Unique association of a Vector $f(t)$, then $f(t)$ is called as a Vector Point Function.

i.e. $(t) = f_1(t)\,\bar{\imath} + f_2(t)\,\bar{\jmath} + f_3(t)\bar{k}$, where $f_1(t)\,, f_2(t)\,, f_3(t)$ are called components of $f(t)$.

Scalar Point Function: Let "D" be a Domain of a function, then if for each variable $t \in D, \exists$ Unique association of a Scalar $f(t)$, then $f(t)$ is called as a Scalar Point Function.

Note: 1) Two Vectors $\bar{a}$ and $\bar{b}$ are said to be Orthogonal (or $\perp^r$) to each other if $\bar{a}\,.\,\bar{b} = 0$

2) Two Vectors $\bar{a}$ and $\bar{b}$ are said to be Parallel to each other if $\bar{a} \times \bar{b} = \bar{0}$.

3) If $\bar{a}, \bar{b}, \bar{c}$ are three vectors, then

a) $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \bar{a}\,.\left(\bar{b} \times \bar{c}\right) = -\bar{a}\,.\left(\bar{c} \times \bar{b}\right)$

b) $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\bar{b}\ \bar{c}\ \bar{a}\right] = \left[\bar{c}\ \bar{a}\ \bar{b}\right] = -\left[\bar{a}\ \bar{c}\ \bar{b}\right] = -\left[\bar{c}\ \bar{b}\ \bar{a}\right]$

c) $\bar{a} \times \left(\bar{b} \times \bar{c}\right) = \left(\bar{a}\,.\,\bar{c}\right)\bar{b} - \left(\bar{a}\,.\,\bar{b}\right)\bar{c}$

4) If $\bar{A}, \bar{B}, \bar{C}$ are three vector point functions over a scalar variable t then

a) $\frac{d}{dt}(\bar{A} + \bar{B}) = \frac{d\bar{A}}{dt} + \frac{d\bar{B}}{dt}$

b) $\frac{d}{dt}(\bar{A}\,.\,\bar{B}) = \frac{d\bar{A}}{dt}\,.\,\bar{B} + \bar{A}\,.\,\frac{d\bar{B}}{dt}$

c) $\frac{d}{dt}(\bar{A} \times \bar{B}) = \frac{d\bar{A}}{dt} \times \bar{B} + \bar{A} \times \frac{d\bar{B}}{dt}$

d) $\frac{d}{dt}\left[\bar{A}\ \bar{B}\ \bar{C}\right] = \left[\frac{d\bar{A}}{dt}\ \ \bar{B}\ \ \bar{C}\right] + \left[\bar{A}\ \ \frac{d\bar{B}}{dt}\ \ \bar{C}\right] + \left[\bar{A}\ \ \bar{B}\ \ \frac{d\bar{C}}{dt}\right]$

e) $\frac{d}{dt}\{\bar{A} \times (B \times \bar{C})\} = \left\{\frac{d\bar{A}}{dt} \times (\bar{B} \times \bar{C})\right\} + \left\{\bar{A} \times \left(\frac{d\bar{B}}{dt} \times \bar{C}\right)\right\} + \left\{\bar{A} \times \left(\bar{B} \times \frac{d\bar{C}}{dt}\right)\right\}$

Constant Vector Function: A Vector Point Function $f(t) = f_1(t)\,\bar{\imath} + f_2(t)\,\bar{\jmath} + f_3(t)\bar{k}$ is said to be constant vector function if $f_1(t)\,, f_2(t)\ \&\ f_3(t)$ are constant.

Note: A Vector Point Function $\bar{f}(t)$ is a constant vector function iff $\frac{d\bar{f}}{dt} = \bar{0}$

- A vector point function $\bar{f}(t)$ has constant magnitude if $\bar{f}\,.\,\frac{d\bar{f}}{dt} = 0$
- A vector point function $\bar{f}(t)$ has constant direction if $\bar{f} \times \frac{d\bar{f}}{dt} = 0$

Vector Differential Operator

The Vector Differential Operator is denoted by ∇ (read as del) and is defined as $\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

i.e. $\nabla \equiv \bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

Now, we define the following quantities which involve the above operator.

- Gradient of a Scalar point function
- Divergence of a Vector point function
- Curl of a Vector point function

Gradient of a Scalar point function

If $\emptyset(x, y, z)$ be any Scalar point function then, the Gradient of $\emptyset$ is denoted by $grad\ \emptyset$ (or) $\nabla\emptyset$, defined as $\nabla\emptyset = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\emptyset$

$$\Rightarrow \nabla\emptyset = \bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}$$

(Or)

Gradient: Let f is a scalar point function, then the gradient of f is denoted by ∇f (or) $grad\ f$ and is defined as $\bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$$\Rightarrow \nabla f = \bar{\imath}\frac{\partial f}{\partial x} + \bar{\jmath}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

Ex: 1) If $f = x^2 + y^2 + z^2$ then $\nabla f = 2x\bar{\imath} + 2y\bar{\jmath} + 2z\bar{k}$

2) If $f = xy + yz + zx$ then $\nabla f = (y + z)\bar{\imath} + (x + z)\bar{\jmath} + (x + y)\bar{k}$

3) If $f = xyz$ then $\nabla f = yx\bar{\imath} + xz\bar{\jmath} + xy\bar{k}$

Note: The Operator gradient is always applied on scalar field and the resultant will be a vector. i.e. The operator gradient converts a scalar field into a vector field.

Properties:

- If f and g are continuous and differentiable scalar point functions then
  - $grad\ (f \pm g) = grad\ f \pm grad\ g$
  - $grad\ \left(\frac{f}{g}\right) = \frac{g\ grad\ f - f\ grad\ g}{g^2}$
  - $grad\ (fg) = g\ grad\ f + f\ grad\ g$

Proof: To Prove $grad\,(f \pm g) = grad\,f \ \pm grad\,g$

Consider L.H.S $grad\,(f \pm g) = \nabla(f \pm g)$

$$= \sum \bar{\imath}\frac{\partial}{\partial x}(f \pm g)$$

$$= \sum \bar{\imath}\left(\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right) = \sum \bar{\imath}\frac{\partial f}{\partial x} \pm \sum \bar{\imath}\frac{\partial g}{\partial x}$$

$$= grad\,f \ \pm grad\,g$$

Similarly, we can prove other results also.

❖ **Show that $d\emptyset = \nabla\emptyset \,.\, d\bar{r}$**

Let $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, then $d\bar{r} = dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k}$.

If $\emptyset$ is any Scalar point function, then $d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$

$$= \left(\bar{\imath}\frac{\partial\emptyset}{\partial x} + \bar{\jmath}\frac{\partial\emptyset}{\partial y} + \bar{k}\frac{\partial\emptyset}{\partial z}\right).\left(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz\right)$$

$$= \nabla\emptyset \,.\, d\bar{r}$$

❖ If $r = |\bar{r}|$ where $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then (1) $\nabla \mathrm{r} = \frac{\bar{r}}{r}$

(2) $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

(3) $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

(4) $\nabla \log r = \frac{\bar{r}}{r^2}$

(5) $\nabla r^n = n\,r^{n-2}\,\bar{r}$

(6) $\nabla f(r) \times \bar{r} = \bar{0}$

Sol: Given that $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $r = |\bar{r}|$

i.e. $r = \sqrt{x^2 + y^2 + z^2} \implies r^2 = x^2 + y^2 + z^2$

Differentiate w.r.t $'\,x\,'$ partially, we get $2\,r\,\frac{\partial r}{\partial x} = 2x \implies \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

(1) $\nabla \mathrm{r} = \sum \bar{\imath}\frac{\partial}{\partial \mathrm{x}}(\mathrm{r}) = \sum \bar{\imath}\left(\frac{x}{r}\right) = \frac{1}{r}\sum \bar{\imath}\,x = \frac{1}{r}\,\bar{r}$

$\therefore\ \nabla \mathrm{r} = \frac{\bar{r}}{r}$

(2) $\nabla f(r) = \sum \bar{\imath}\,\frac{\partial}{\partial x}f(r) = \sum \bar{\imath}\ f'(r)\,\frac{\partial r}{\partial x}$

$= f'(r)\sum \bar{\imath}\,\frac{\partial r}{\partial x} = f'(r)\sum \bar{\imath}\,\frac{x}{r}$

$= f'(r)\,\frac{\bar{r}}{r}$

Here ∇ means Differentiation

(3) $\nabla\left(\frac{1}{r}\right) = \nabla(r^{-1}) = \sum \bar{\iota}\frac{\partial}{\partial \mathrm{x}}(\mathrm{r})^{-1} = -\sum \bar{\iota}\, r^{-2}\left(\frac{x}{r}\right) = -\frac{\bar{r}}{r^3}$

(4) Similarly, we can prove this

(5) Similarly, we can prove this

(6) $\nabla f(r) \times \bar{r} = \left(f'(r)\ \frac{\bar{r}}{r}\right) \times \bar{r}$

$$= \frac{f'(r)}{r}\,(\bar{r} \times \bar{r}) = \bar{0}$$

Note: If $\emptyset$ is any scalar point function (surface), then Normal Vector along $\emptyset$ is given by $\nabla\emptyset$ (or) $grad\ \emptyset$ and Unit Normal Vector along $\emptyset$ is $\frac{\nabla\emptyset}{|\nabla\emptyset|}$

Directional Derivative

The directional derivative of a scalar point function $\emptyset$ at point P in the direction of a vector point function $\bar{f}$ is given by $\boldsymbol{e}\,.\,(\boldsymbol{\nabla\emptyset})_{\boldsymbol{P}}$, where e is unit vector along $\bar{f}$.

i.e. $e = \frac{\bar{f}}{|\bar{f}|}$

Note: If $\emptyset$ is any scalar point function, then along the direction of $\nabla\emptyset$, the directional derivative of $\emptyset$ is maximum, and also the maximum value of directional derivative of $\emptyset$ at point P is given by $|\nabla\emptyset|_P$.

- $\nabla\emptyset$ is parallel to x-axis $\Longrightarrow$ Co-efficient of $\bar{\jmath} = 0$ and Co-efficient of $\bar{k} = 0$
- $\nabla\emptyset$ is parallel to y-axis $\Longrightarrow$ Co-efficient of $\bar{\iota} = 0$ and Co-efficient of $\bar{k} = 0$
- $\nabla\emptyset$ is parallel to z-axis $\Longrightarrow$ Co-efficient of $\bar{\iota} = 0$ and Co-efficient of $\bar{\jmath} = 0$

Angle Between two surfaces

If θ is the angle between the two surfaces $\emptyset_1$ and $\emptyset_2$, then $\cos\theta = \frac{\nabla\emptyset_1\,.\,\nabla\emptyset_2}{|\nabla\emptyset_1|\,|\nabla\emptyset_2|}$

Angle between two vectors

- The angle between the two vectors $\bar{A}$ & $\bar{B}$ is given by $\cos\theta = \frac{\bar{A}\,.\,\bar{B}}{|\bar{A}|\,|\bar{B}|}$

Note: If $\bar{r} = x\bar{\iota} + y\bar{\jmath} + z\bar{k}$ be position vector along any vector where x, y, z are in terms of scalar t, then $\frac{d\bar{r}}{dt}$ gives velocity and $\frac{d^2r}{dt^2}$ gives acceleration.

- If $\bar{V}$ is velocity of a particle, then the component of velocity in the direction of $\bar{A}$ is given by $\frac{\bar{V}\,.\,\bar{A}}{|\bar{A}|}$

❖ If $\bar{A}$ is acceleration of a particle then the component of acceleration in the direction of $\bar{B}$ is given by $\frac{\bar{A}\,.\,\bar{B}}{|\bar{B}|}$

Projection of a Vector

❖ The Projection of a vector $\bar{A}$ on $\bar{B}$ is $\frac{\bar{A}\,.\,\bar{B}}{|\bar{B}|}$

❖ The Projection of a vector $\bar{B}$ on $\bar{A}$ is $\frac{\bar{A}\,.\,\bar{B}}{|\bar{A}|}$

Divergent: Let $\bar{f} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is a vector point function, then the divergent of f is denoted by $\nabla.f$ (or) $div\ \bar{f}$ and is defined as $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$\Rightarrow \ \nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Ex: 1) If $\bar{f} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $\nabla.\bar{f} = 1 + 1 + 1 = 3$

2) If $\bar{f} = xy\bar{\imath} + yz\bar{\jmath} + zx\bar{k}$ then $\nabla.\bar{f} = y + z + x$

If we substitute x, y, z values, then we get vector point function

Note: 1) The Operator divergent is always applied on a vector field, and the resultant will be a scalar.

I.e. The operator divergent will converts a vector into a scalar.

2) Divergent of a constant vector is always zero

Ex: $\bar{f} = 2\bar{\imath} + 3\bar{\jmath} + 4\bar{k}$ then $\nabla.\bar{f} = 0$.

Solenoidal Vector: If $div\ \bar{f} = 0$, then $\bar{f}$ is called as Solenoidal vector.

Ex: If $\bar{f} = x^2\bar{\imath} - xy\bar{\jmath} - xz\bar{k} \ \Rightarrow div\ \bar{f} = 0$

$\therefore\ \bar{f}$ is Solenoidal vector.

Curl of a Vector: Let $\bar{f} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is a vector valued function, then curl of vector f is denoted by $curl\ \bar{f}$ and is defined as $\nabla \times \bar{f}$

$$\therefore curl\ f = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Ex: 1) If $\bar{f} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{0}$

2) If $\bar{f} = x^2\bar{\imath} + y^2\bar{\jmath} + z^2\bar{k}$ then $curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = (xz - y^2)\bar{\imath} - yz\,\bar{\jmath}$

Note: The Operator ***curl*** is applied on a vector field.

Irrotational Vector: The vector $\bar{f}$ is said to be Irrotational if $curl\ \bar{f} = \bar{0}$.

Ex: If $\bar{f} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $curl\ \bar{f} = \nabla \times \bar{f} = \bar{0}$

$\Rightarrow$ $\bar{f}$ is called as Irrotational Vector.

Note: 1) If $curl(grad\ \emptyset) = \bar{0} \Rightarrow grad\ \emptyset$ is always an Irrotational Vector.

2) If $div(curl\ \bar{f}) = 0 \Rightarrow curl\ \bar{f}$ is always Solenoidal Vector.

Theorems

1) If $\boldsymbol{\phi}$ is any scalar point function and $'\bar{a}'$ is a vector point function , then

$$\boldsymbol{div(\phi\bar{a}) = (grad\ \phi).\bar{a} + \phi\ div\ \bar{a}}$$

(Or)

$$\boldsymbol{\nabla.(\phi\bar{a}) = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})}$$

Sol: To Prove $div(\phi\bar{a}) = (grad\ \phi).\bar{a} + \phi\ div\ \bar{a}$

Consider $div(\phi\bar{a}) = \sum \bar{\imath}.\frac{\partial}{\partial x}(\phi\bar{a})$

$$= \sum \bar{\imath}.\left(\frac{\partial\phi}{\partial x}\ \bar{a} + \phi\frac{\partial a}{\partial x}\right)$$

$$= \sum\left(\bar{\imath}\frac{\partial\emptyset}{\partial x}\right).\bar{a} + \emptyset\sum\bar{\imath}.\frac{\partial\bar{a}}{\partial x}$$

$$= (grad\ \phi).\bar{a} + \phi\ div\ \bar{a}$$

2) If $\boldsymbol{\phi}$ is any scalar point function and $'\bar{a}'$ is a vector point function , then

$$\boldsymbol{curl\ (\phi\bar{a}) = (grad\ \phi) \times \bar{a} + \phi\ curl\ \bar{a}}$$

(or)

$$\boldsymbol{\nabla \times (\phi\bar{a}) = (\nabla\phi) \times \bar{a} + \phi(\nabla \times \bar{a})}$$

Sol: To Prove $curl\ (\phi\bar{a}) = (grad\ \phi) \times \bar{a} + \phi\ curl\ \bar{a}$

Consider $curl\ (\phi\bar{a}) = \sum \bar{\imath} \times \frac{\partial}{\partial x}(\phi\bar{a})$

$$= \sum \bar{\imath} \times \left(\frac{\partial\phi}{\partial x}\ \bar{a} + \phi\frac{\partial a}{\partial x}\right)$$

$$= \sum\left(\bar{\imath}\frac{\partial\emptyset}{\partial x}\right) \times \bar{a} + \emptyset\sum\bar{\imath} \times \frac{\partial\bar{a}}{\partial x} \quad = (grad\ \phi) \times \bar{a} + \phi\ curl\ \bar{a}$$

3) If $\bar{A}$ and $\bar{B}$ are two vector point functions then,

$$\boldsymbol{grad\ (\bar{A}.\bar{B}) = (B.\nabla)\bar{A} + (A.\nabla)\bar{B} + \bar{B} \times curl\ \bar{A} + \bar{A} \times curl\ \bar{B}}$$

Sol: Let us consider R.H.S

In R.H.S consider $\bar{A} \times curl\ \bar{B} = \bar{A} \times \sum\left(\bar{\imath} \times \frac{\partial \bar{B}}{\partial x}\right)$

Since $\bar{A}$ is a vector, take it inside summation.

$$= \sum \bar{A} \times \left(\bar{\imath} \times \frac{\partial \bar{B}}{\partial x}\right)$$

$$= \sum\left\{\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath} - (\bar{A}.\bar{\imath})\frac{\partial \bar{B}}{\partial x}\right\}$$

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

$$= \sum\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath} - \sum(\bar{A}.\bar{\imath})\frac{\partial \bar{B}}{\partial x}$$

$$= \sum\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath} - \sum\left(\bar{A}.\bar{\imath}\frac{\partial}{\partial x}\right)\bar{B}$$

$$\therefore\ \bar{A} \times curl\ \bar{B} = \sum\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath} - (\bar{A}.\nabla)\bar{B}$$

Similarly, $\bar{B} \times curl\ \bar{A} = \sum\left(\bar{B}.\frac{\partial \bar{A}}{\partial x}\right)\bar{\imath} - (\bar{B}.\nabla)\bar{A}$

Now, R.H.S $(B.\nabla)\bar{A} + (A.\nabla)\bar{B} + \bar{B} \times curl\ \bar{A} + \bar{A} \times curl\ \bar{B}$

$$= (B.\nabla)\bar{A} + (A.\nabla)\bar{B} + \sum\left(\bar{B}.\frac{\partial \bar{A}}{\partial x}\right)\bar{\imath} - (\bar{B}.\nabla)\bar{A} + \sum\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath} - (\bar{A}.\nabla)\bar{B}$$

$$= \sum\left(\bar{B}.\frac{\partial \bar{A}}{\partial x}\right)\bar{\imath} + \sum\left(\bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath}$$

$$= \sum\left(\bar{B}.\frac{\partial \bar{A}}{\partial x} + \bar{A}.\frac{\partial \bar{B}}{\partial x}\right)\bar{\imath}$$

$$= \sum \bar{\imath}\frac{\partial}{\partial x}(\bar{A}.\bar{B}) = grad\ (\bar{A}.\bar{B})$$

4) Prove that $\boldsymbol{div\ (\bar{A} \times \bar{B}) = \bar{B}.curl\ \bar{A} - \bar{A}.curl\ \bar{B}}$

Sol: *Consider L.H.S:* $div\ (\bar{A} \times \bar{B}) = \sum \bar{\imath}.\frac{\partial}{\partial x}(\bar{A} \times \bar{B})$

$$= \sum \bar{\imath}.\left(\frac{\partial \bar{A}}{\partial x} \times \bar{B} + \bar{A} \times \frac{\partial \bar{B}}{\partial x}\right)$$

$$= \sum \bar{\imath}.\left(\frac{\partial \bar{A}}{\partial x} \times \bar{B}\right) + \sum \bar{\imath}.\left(\bar{A} \times \frac{\partial \bar{B}}{\partial x}\right)$$

We know that $\bar{a}.(\bar{b} \times \bar{c}) = \bar{b}.(\bar{c} \times \bar{a}) = \bar{c}.(\bar{a} \times \bar{b})$ *and* $(\bar{a} \times \bar{b}) = -(\bar{b} \times \bar{a})$

$$= \sum \bar{B}.\left(\bar{\imath} \times \frac{\partial \bar{A}}{\partial x}\right) - \sum \bar{\imath}.\left(\frac{\partial \bar{B}}{\partial x} \times \bar{A}\right)$$

$$= \bar{B}.curl\ \bar{A} - \sum \bar{A}.\left(\bar{\imath} \times \frac{\partial \bar{B}}{\partial x}\right)$$

$$= \bar{B}.curl\ \bar{A} - \bar{A}.curl\ \bar{B}$$

$$\therefore\ \boldsymbol{div\ (\bar{A} \times \bar{B}) = \bar{B}.curl\ \bar{A} - \bar{A}.curl\ \bar{B}}$$

5) *Prove that curl* $\left(\bar{\boldsymbol{a}}\times\bar{\boldsymbol{b}}\right)=\bar{\boldsymbol{a}}\,\boldsymbol{div}\bar{\boldsymbol{b}}-\bar{\boldsymbol{b}}\,\boldsymbol{div}\bar{\boldsymbol{a}}+\left(\bar{\boldsymbol{b}}\,.\nabla\right)\bar{\boldsymbol{a}}-\left(\bar{\boldsymbol{a}}\,.\nabla\right)\bar{\boldsymbol{b}}$

Sol: Consider L.H.S: $curl\left(\bar{a}\times\bar{b}\right)=\sum\bar{\imath}\times\frac{\partial}{\partial x}\left(\bar{a}\times\bar{b}\right)$

$$=\sum\bar{\imath}\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b}+\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$=\sum\bar{\imath}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)+\sum\bar{\imath}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

We know that $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=(\bar{a}.\bar{c})\bar{b}-\left(\bar{a}.\bar{b}\right)\bar{c}$

$$\Rightarrow curl\left(\bar{a}\times\bar{b}\right)=\sum\left\{\left(\bar{\imath}.\bar{b}\right)\frac{\partial\bar{a}}{\partial x}-\left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\}+\sum\left\{\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-(\bar{\imath}.\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$=\sum\left(\bar{b}.\bar{\imath}\right)\frac{\partial\bar{a}}{\partial x}-\sum\left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-\sum(\bar{a}\,.\bar{\imath})\frac{\partial\bar{b}}{\partial x}$$

$$=\sum\left(\bar{b}.\bar{\imath}\frac{\partial}{\partial x}\right)\bar{a}-\sum\left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-\sum\left(\bar{a}.\bar{\imath}\frac{\partial}{\partial x}\right)\bar{b}$$

$$=\sum\left(\bar{b}.\bar{\imath}\frac{\partial}{\partial x}\right)\bar{a}-\sum\left(\bar{\imath}.\frac{\partial\bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{\imath}.\frac{\partial\bar{b}}{\partial x}\right)\bar{a}-\left(\bar{a}.\sum\bar{\imath}\frac{\partial}{\partial x}\right)\bar{b}$$

$$=\left(\bar{b}.\nabla\right)\bar{a}-(\nabla.\bar{a})\bar{b}+\left(\nabla.\bar{b}\right)\bar{a}-(\bar{a}\,.\nabla)\bar{b}$$

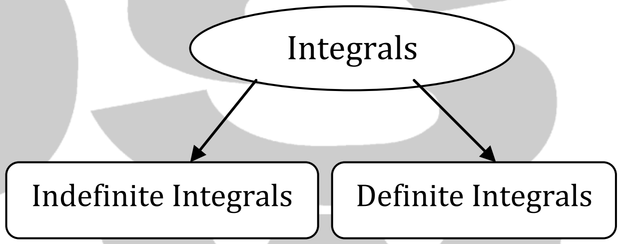
$$\therefore\ \boldsymbol{curl}\left(\bar{\boldsymbol{a}}\times\bar{\boldsymbol{b}}\right)=\bar{\boldsymbol{a}}\,\boldsymbol{div}\bar{\boldsymbol{b}}-\bar{\boldsymbol{b}}\,\boldsymbol{div}\bar{\boldsymbol{a}}+\left(\bar{\boldsymbol{b}}\,.\nabla\right)\bar{\boldsymbol{a}}-\left(\bar{\boldsymbol{a}}\,.\nabla\right)\bar{\boldsymbol{b}}$$

Vector Integration

Integration is the inverse operation of differentiation.

Integrations are of two types. They are

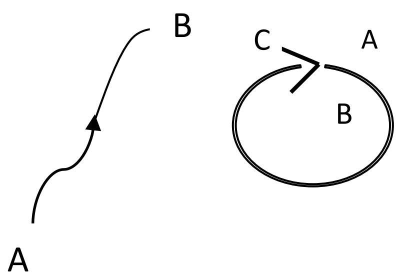
1) Indefinite Integral
2) Definite Integral



Line Integral

Any Integral which is evaluated along the curve is called Line Integral, and it is denoted by $\int_c \bar{F}.d\bar{r}$ where $\bar{F}$ is a vector point function, $\bar{r}$ is position vector and C is the curve.

Let C be a curve in space. Let A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive, then the direction from B to A is called negative direction. If the two points A and B coincide the curve C is called the closed curve.



Note: 1) If $\bar{r}$ is given in terms of $(x\,,y\,,z)$

Let $\bar{F}=f_1\bar{\imath}+f_2\bar{\jmath}+f_3\bar{k},\bar{r}=x\bar{\imath}+y\bar{\jmath}+z\bar{k}$

then $d\bar{r}=dx\bar{\imath}+dy\bar{\jmath}+dz\bar{k}$

$\Rightarrow\ \bar{F}.d\bar{r}=f_1dx+f_2dy+f_3dz$

This is used when $\bar{r}$ is given in terms of $(x\,,y\,,z)$

$\therefore \int_C \bar{F}.d\bar{r} = \int_C (f_1 dx + f_2 dy + f_3 dz)$

2) If $\bar{r}$ is given in terms of $'\, t\, '$

Let $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$, where f_1, f_2, f_3 are functions of $(x\,, y\,, z)$ and

$\boldsymbol{r} = \phi_1(t)\bar{\imath} + \phi_2(t)\bar{\jmath} + \phi_3(t)\bar{k}$ then $\int_C \bar{F}.d\bar{r} = \int_C \left(\bar{F}.\frac{dr}{dt}\right) dt$

Surface Integral

The Integral which is evaluated over a surface is called Surface Integral.

If S is any surface and N is the outward drawn unit normal vector to the surface S then $\int_S \bar{F}.N\ ds$ is called the Surface Integral.

Note: Let $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ and $N = \bar{\imath}\cos\alpha + \bar{\jmath}\cos\beta + \bar{k}\cos\gamma$

$$\bar{F}.N = f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma$$

$$\int_S \bar{F}.N\ ds = \int_S (f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma\,)\,ds$$

$$= \int_S (f_1\cos\alpha\, ds + f_2\cos\beta\, ds + f_3\cos\gamma\ ds)$$

$$= \int_S (f_1 dy\, dz + f_2 dx\, dz + f_3 dx\, dy)$$

Here, $ds\cos\gamma = dx\, dy$

$ds\cos\beta = dx\, dz$

$ds\cos\alpha = dy\, dz$

Now,

- $ds\cos\gamma = dx\, dy$

$\Rightarrow ds = \dfrac{dx\, dy}{\cos\gamma} = \dfrac{dx\, dy}{N.\bar{k}}$

- $ds\cos\beta = dx\, dz$

$\Rightarrow ds = \dfrac{dx\, dz}{\cos\beta} = \dfrac{dx\, dz}{N.\bar{\jmath}}$

- $ds\cos\alpha = dy\, dz$

$\Rightarrow ds = \dfrac{dy\, dz}{\cos\alpha} = \dfrac{dy\, dz}{N.\bar{\imath}}$

- If R is the projection of S lies on xy plane then $\int_S \bar{F}.N\ ds = \iint_S \bar{F}.N\ \frac{dx\, dy}{|N.\bar{k}|}$
- If R is the projection of S lies on xy plane then $\int_S \bar{F}.N\ ds = \iint_S \bar{F}.N\ \frac{dx\, dz}{|N.\bar{\jmath}|}$
- If R is the projection of S lies on xy plane then $\int_S \bar{F}.N\ ds = \iint_S \bar{F}.N\ \frac{dy\, dz}{|N.\bar{\imath}|}$

Note: No one will say/ guess directly by seeing the problem that projection will lies on xy-plane (or) yz-plane (or) zx- plane for a particular problem. It mainly depends on N.

Note: In solving surface integral problems (mostly)

- If given surface is in xy-plane, then take projection on yz-plane.
- If given surface is in xz-plane, then take projection on yz-plane.

- If given surface is in yz-plane, then take projection on xy-plane.

Volume Integral

- If $\bar{F}$ is a vector point function bounded by the region R with volume V, then $\int_V \bar{F}\ dV$ is called as Volume Integral.

  i.e. If $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ then $\int_V \bar{F}\ dV = \int_V (f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k})\, dV$

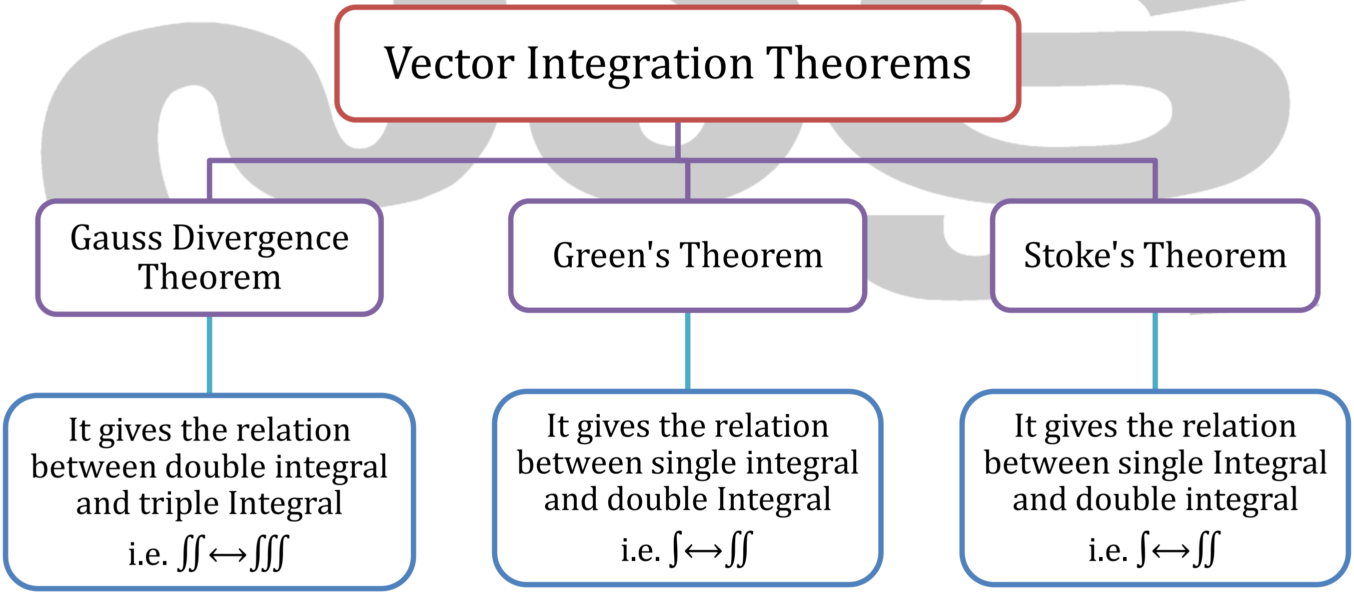
$$= \int_x \int_y \int_z (f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k})\ dV$$

$$= [\iiint f_1 dz\, dy\, dx]\,\bar{\imath} + [\iiint f_2 dz\, dy\, dx]\,\bar{\jmath} + [\iiint f_3 dz\, dy\, dx]\,\bar{k}$$

- If $f = f(x, y, z)$ be any scalar point function bounded by the region R with volume V, then $\int_V f\ dV$ is called as Volume Integral.

i.e. If $f = f(x, y, z)$ then $\int_V f\ dV = \iiint f\ \ dz\, dy\, dx$

Vector Integration



Why these theorems are used?

While evaluating Integration (single/double/triple) problems, we come across some Integration problems where evaluating single integration is too hard, but if we change the same problem in to double integration, the Integration problem becomes simple. In such cases, we use Greens

Theorem (if the given surface is xy-plane) (or) Stokes Theorem (for any plane). If we want to change double integration problem in to triple integral, we use Gauss Divergence Theorem.

- Greens Theorem is used if the given surface is in xy-plane only.
- Stokes Theorem is used for any surface (or) any plane (xy-plane, yz-plane, zx-plane)

Green's Theorem

Let S be a closed region in xy-plane bounded by a curve C. If $P(x, y)$ and $Q(x, y)$ be the two continuous and differentiable Scalar point functions in (x, y) then

$$\boldsymbol{\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Note: This theorem is used if the surface is in xy-plane only.

This Theorem converts single integration problem to double integration problem.

Gauss Divergence Theorem

Let S is a closed surface enclosing a volume V, if $\bar{F}$ is continuous and differentiable vector point function the

$$\boldsymbol{\int_S \bar{F}\,.\,\bar{N}\,ds = \int_V div\,\bar{F}\,\,dV}$$

Where $\bar{N}$ is the outward drawn Unit Normal Vector.

Note: This theorem is used to convert double integration problem to triple integration problem

Stokes Theorem

Let S is a surface enclosed by a closed curve C and $\bar{F}$ is continuous and differentiable vector point function then

$$\int_C \bar{F}.d\bar{r} = \int_S curl\,\bar{F}\,.\,\bar{N}\,ds$$

Where, $\bar{N}$ is outward drawn Unit Normal Vector over S.

Note: This theorem is used for any surface (or) plane.

This theorem is used to convert single integration problem to triple integration problem.

Important: Changing Co-ordinates from one to another

1) **Cylindrical Co-ordinates**

In order to change co-ordinates from Cartesian to Cylindrical, the adjacent values are to be taken, which will be helpful in solving problems in Gauss Divergence Theorem.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$dv = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$\left.\begin{array}{l}\theta = 0\ to\ 2\pi \\ r = 0\ to\ a\end{array}\right\} \text{Always Same}$$

$z = 0\ to\ 3 \longrightarrow$ varies

Here $'a'$ is given radius

2) **Polar Co-ordinates**

In order to change co-ordinates from Cartesian to Polar, the adjacent values are to be taken, which will be helpful in solving problems in Gauss Divergence Theorem.

$For\ x^2 + y^2 = r^2$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta$$

Here, $r = 0\ to\ given$ and $\theta = 0\ to\ 2\pi$

3) **Spherical Co-ordinates**

In order to change co-ordinates from Cartesian to Spherical, the adjacent values are to be taken, which will be helpful in solving problems in Gauss Divergence Theorem.

$$x = r\sin\theta\cos\emptyset$$
$$y = r\sin\theta\sin\emptyset$$
$$z = r\cos\theta$$
$$dv = dx\,dy\,dz = r^2\,dr\,d\theta\,dz$$

$r = 0\ to\ a$ (Changes)

$$\left.\begin{array}{l}\theta = 0\ to\ \frac{\pi}{2} \\ \emptyset = 0\ to\ 2\pi\end{array}\right\} \longrightarrow \text{No change}$$

* * *