

# EELE 3331 – Electromagnetic I

## Chapter 3

# Vector Calculus

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# Differential Length, Area, and Volume

This chapter deals with integration and differentiation of vectors

→ Applications: Next Chapter.

## Differential Length, Area, and Volume

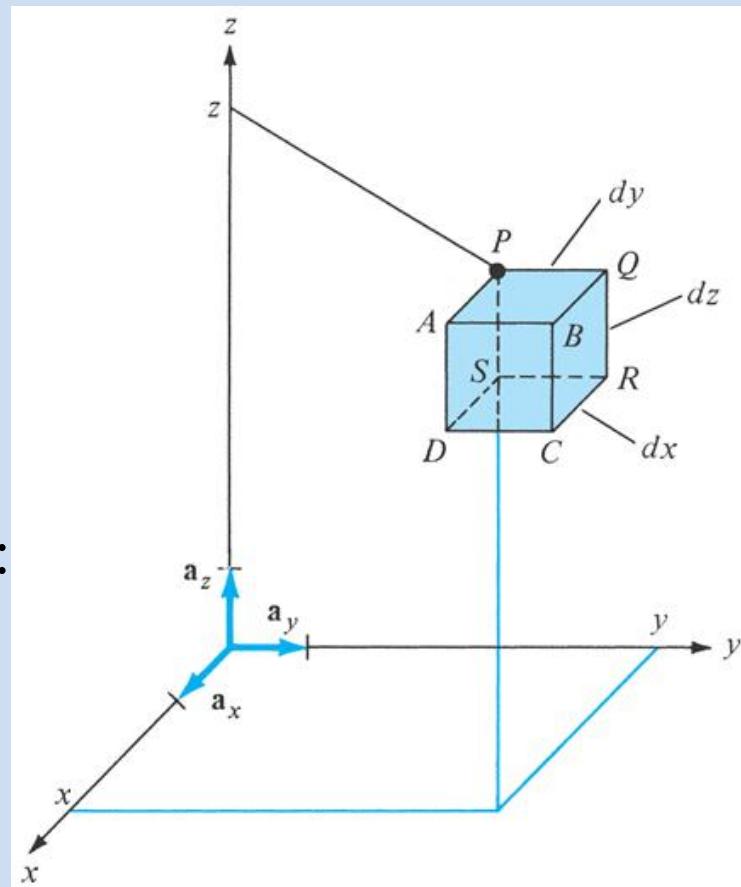
### A. Cartesian Coordinate Systems:

#### 1. Differential displacement:

Differential displacement from point

$S(x,y,z)$  to point  $B(x+dx, y+dy, z+dz)$  is:

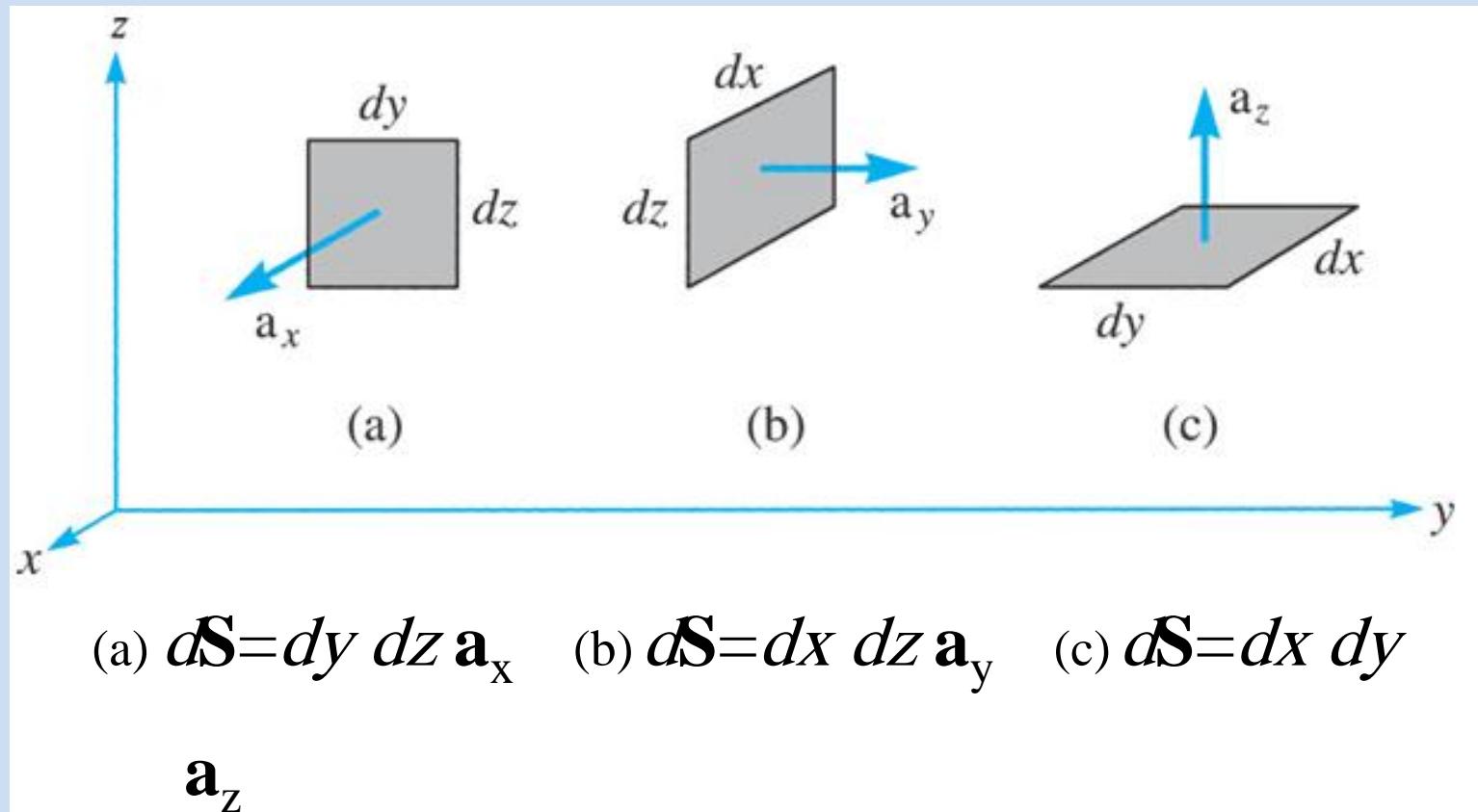
$$d\mathbf{l} = dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z$$



# Differential Length, Area, and Volume

## A. Cartesian Coordinate Systems:

### 2. Differential normal surface area



# Differential Length, Area, and Volume

## A. Cartesian Coordinate Systems:

### 3. Differential volume

$$dV = dx \, dy \, dz$$

Notes:

$d$ ,  $d\mathbf{S}$  → Vectors

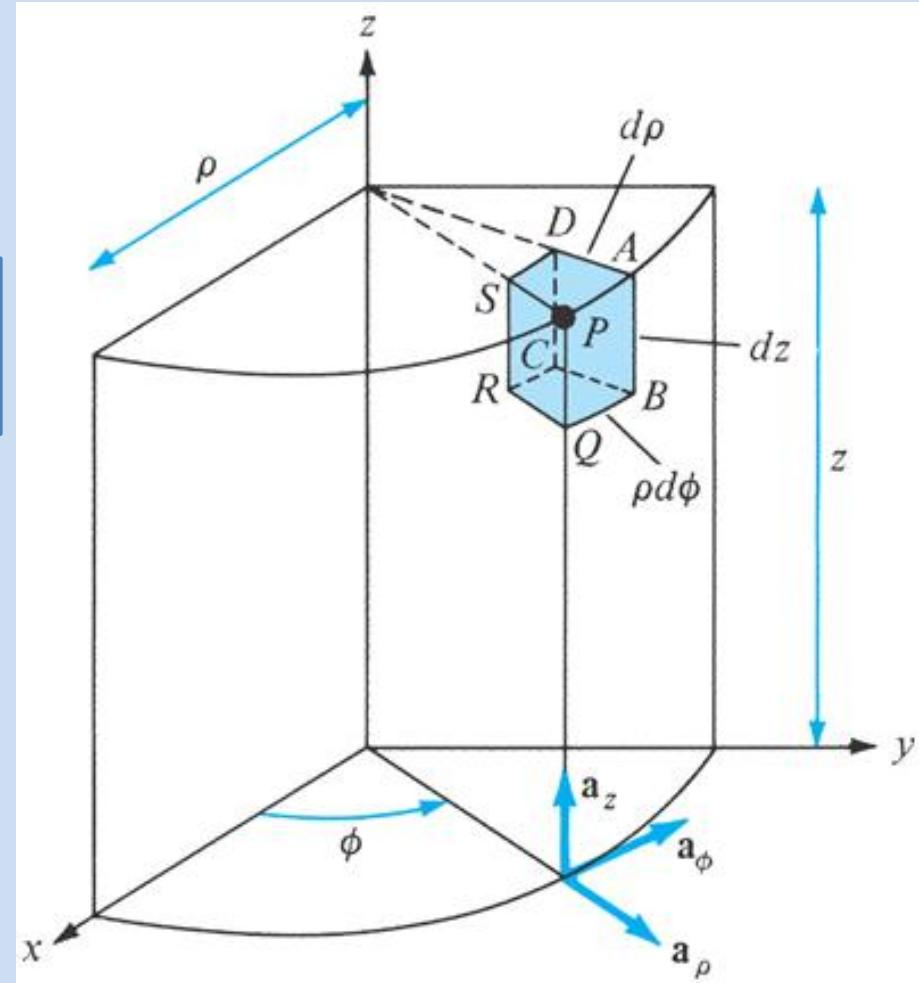
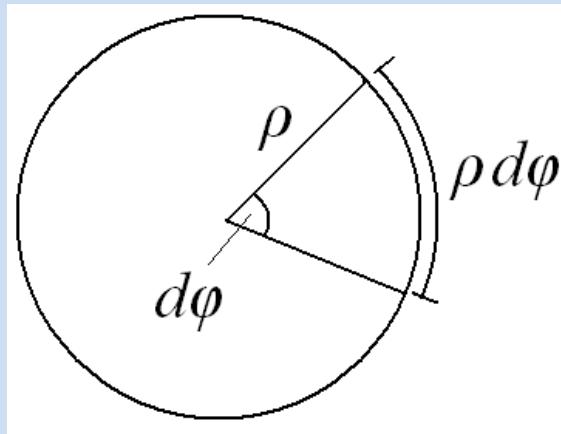
$dV$  → Scalar

# Differential Length, Area, and Volume

## B. Cylindrical Coordinate Systems:

### 1. Differential displacement:

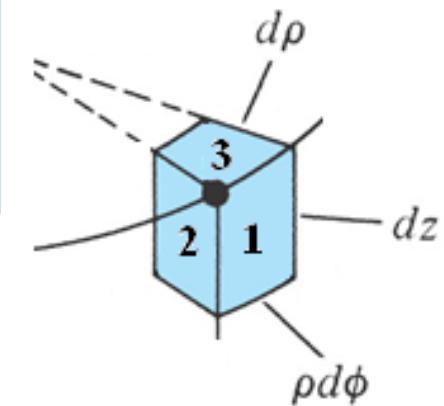
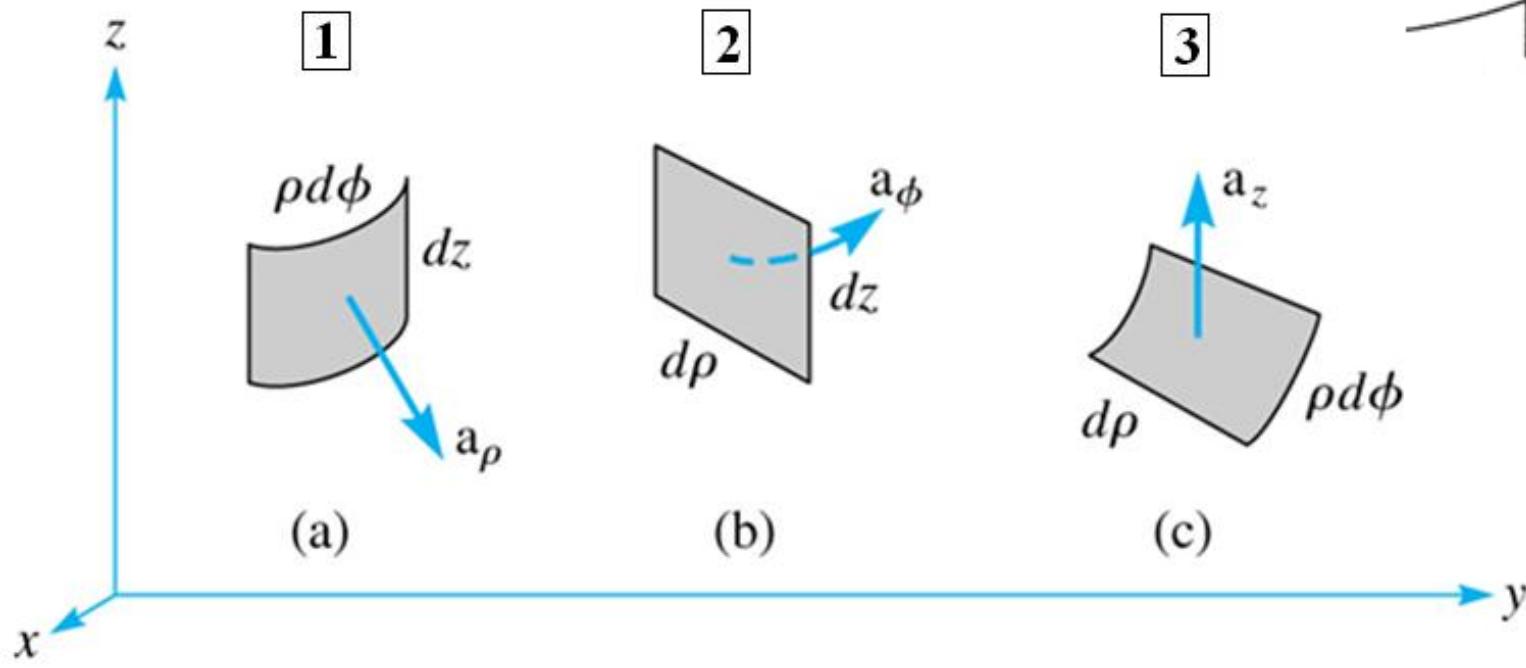
$$d\mathbf{l} = d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi + dz \mathbf{a}_z$$



# Differential Length, Area, and Volume

## B. Cylindrical Coordinate Systems:

### 2. Differential normal surface area



Note:  $d\mathbf{S}$   
can be  
derived  
from  $dV$

$$(a) d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$$

$$(b) d\mathbf{S} = d\rho dz \mathbf{a}_\phi$$

$$(c) d\mathbf{S} = \rho d\rho d\phi \mathbf{a}_z$$

### 3. Differential volume

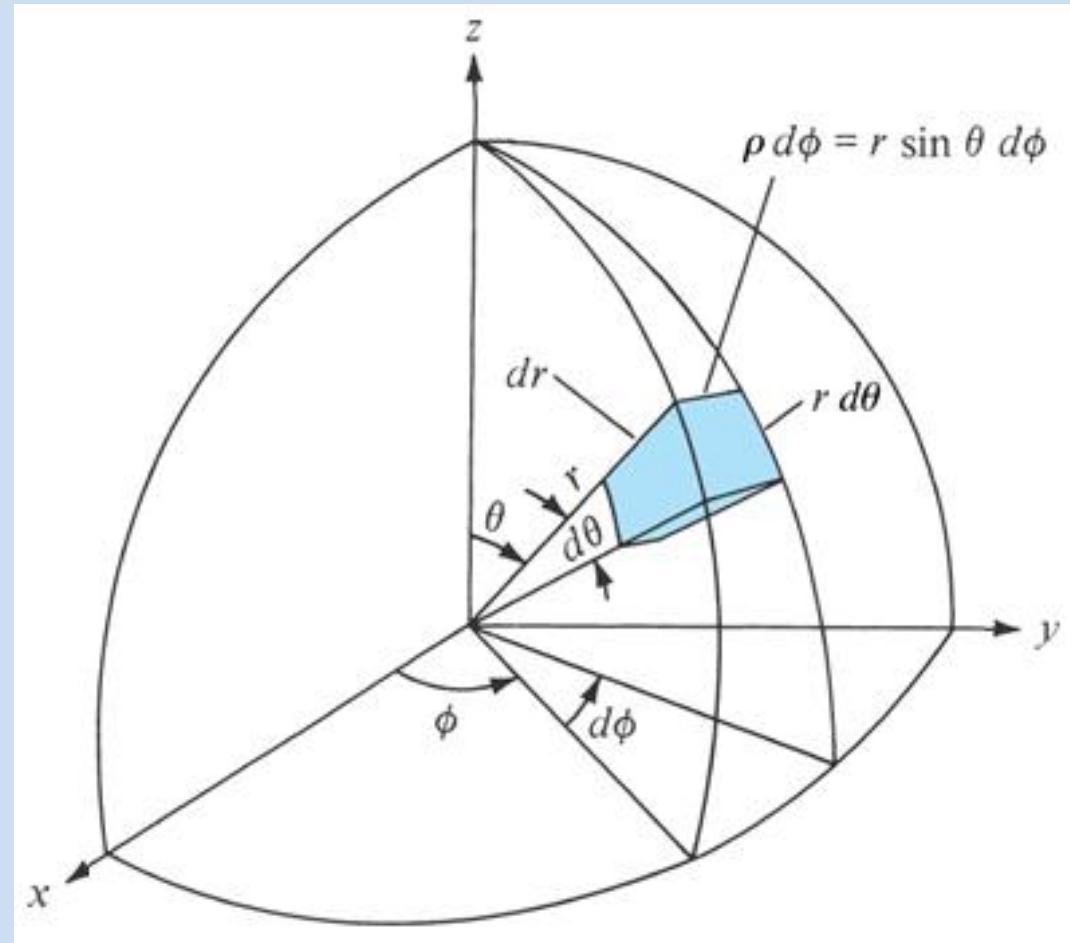
$$dV = \rho d\rho d\phi dz$$

# Differential Length, Area, and Volume

## C. Spherical Coordinate Systems:

### 1. Differential displacement:

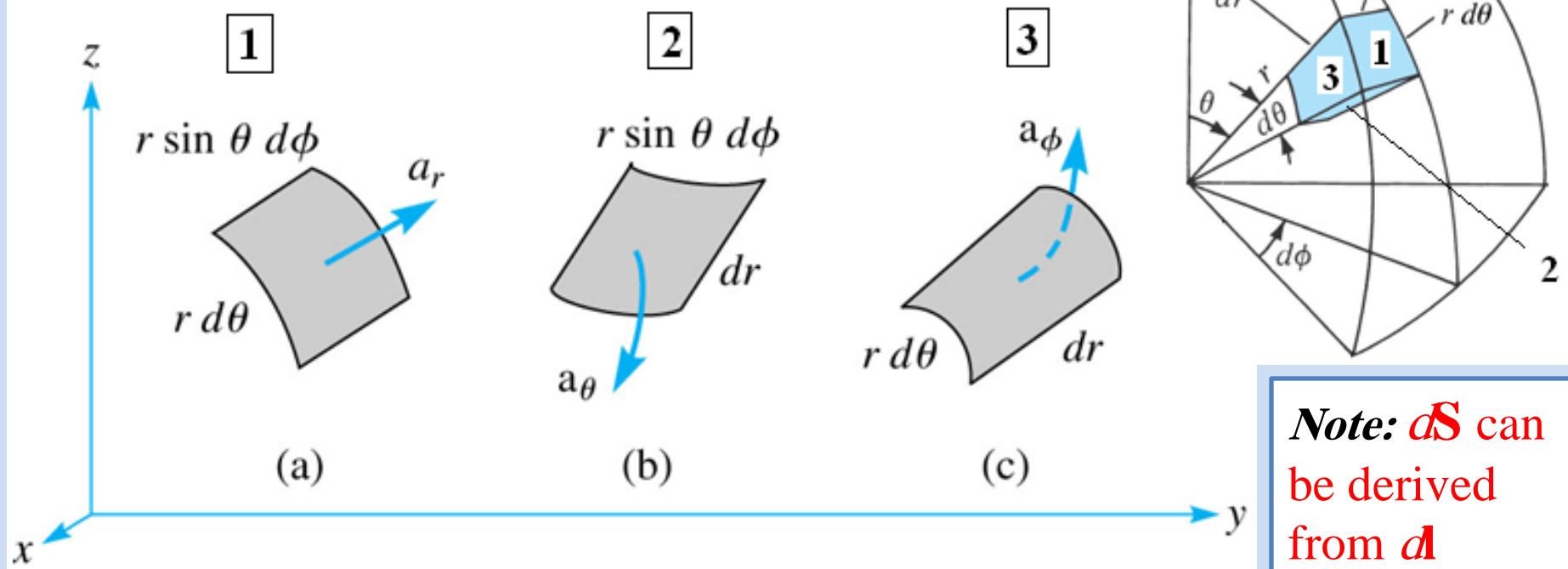
$$\begin{aligned}\mathbf{d} = & dr \mathbf{a}_r \\ & + r d\theta \mathbf{a}_\theta \\ & + r \sin \theta d\phi \mathbf{a}_\phi\end{aligned}$$



# Differential Length, Area, and Volume

## C. Spherical Coordinate Systems:

### 2. Differential normal surface area



$$(a) d\mathbf{S} = r^2 \sin \theta d\theta d\phi \mathbf{a}_r \quad (b) d\mathbf{S} = r \sin \theta dr d\phi \mathbf{a}_\theta \quad (c) d\mathbf{S} = r dr d\theta \mathbf{a}_\phi$$

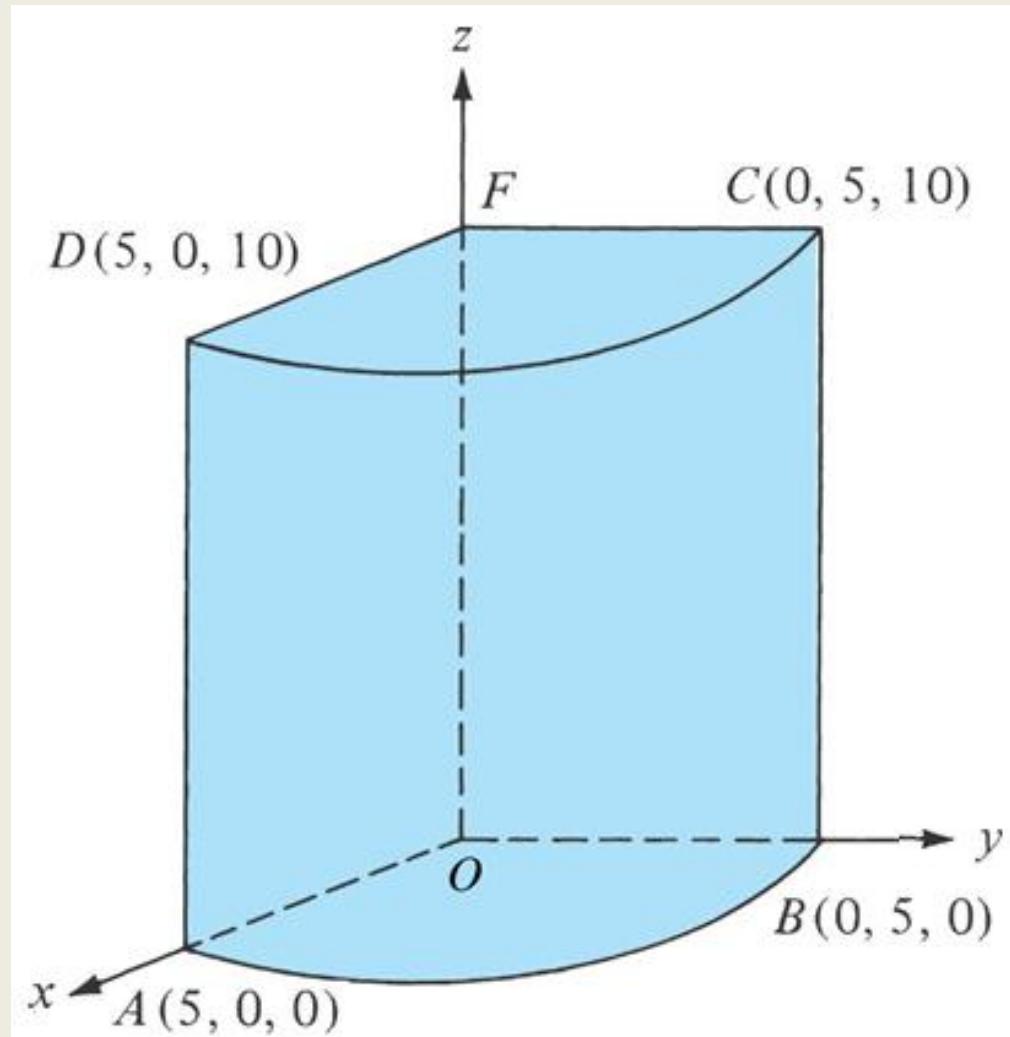
### 3. Differential volume

$$dv = r^2 \sin \theta dr d\theta d\phi$$

## Example 3.1

Consider the object shown. Calculate :

- (a) The length BC
- (b) The length CD
- (c) The surface area ABCD
- (d) The surface area ABO
- (e) The surface area AOFD
- (f) The volume ABCDFO



## Example 3.1 - solution

Object has Cylindrical Symmetry  $\rightarrow$  Cylindrical Coordinates

Cartesian to Cylindrical:

$$A(5,0,0) \rightarrow A(5,0^0,0)$$

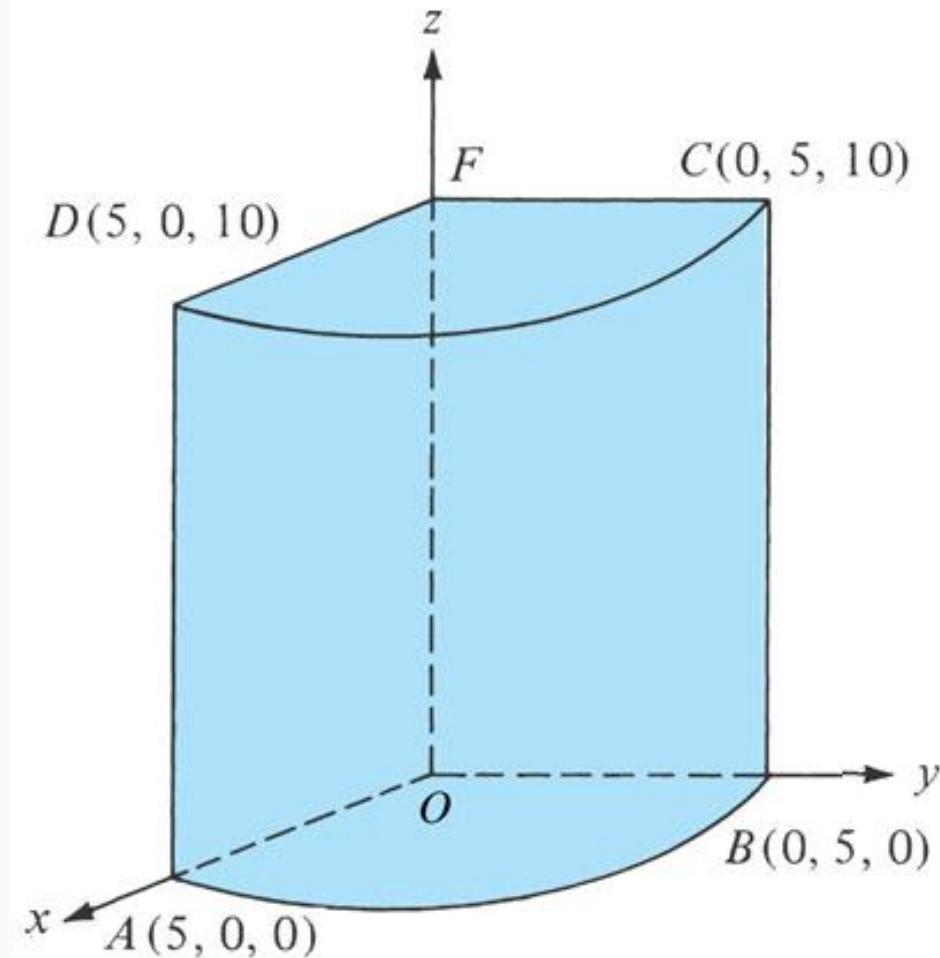
$$B(0,5,0) \rightarrow B(5,\pi/2,0)$$

$$C(0,5,10) \rightarrow C(5,\pi/2,10)$$

$$D(5,0,10) \rightarrow D(5,0^0,10)$$

(a) along BC,  $dl = dz$

$$BC = \int dl = \int_0^{10} dz = 10$$



(b) Along CD,  $dl = \rho d\phi \rightarrow CD = \int_0^{\pi/2} \rho d\phi = 5\phi \Big|_0^{\pi/2} = 2.5\pi$

## Example 3.1 - solution

(c) for ABCD,  $dS = \rho d\phi dz$ ,  $\rho=5$

$$\text{Area } ABCD = \int dS$$

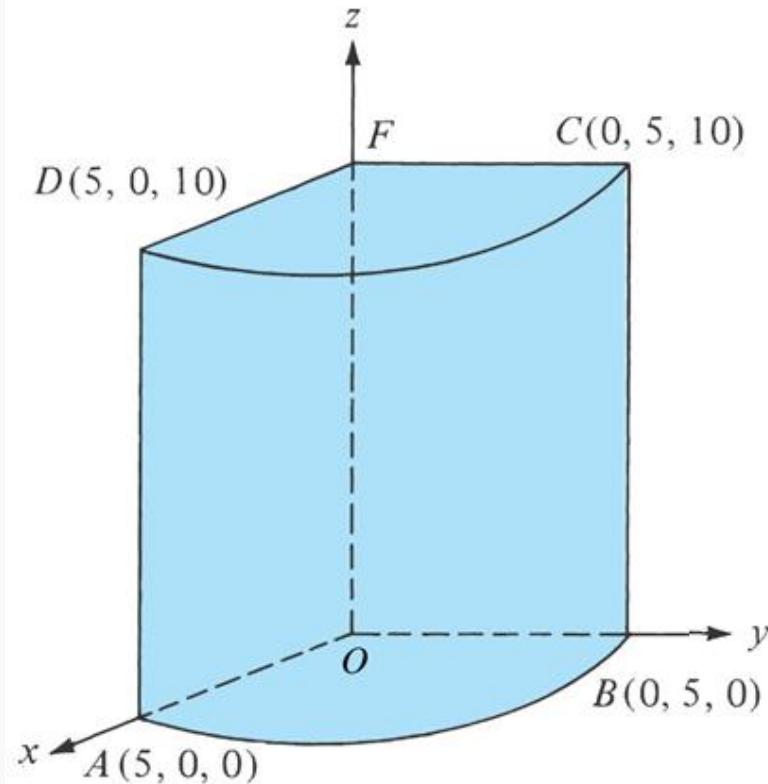
$$= \int_{z=0}^{10} \int_{\phi=0}^{\pi/2} \rho d\phi dz = (5)(\pi/2)(10)$$

$$= 25\pi$$

(d) for ABO,  $dS=\rho d\phi d\rho$ ,  $z=0$

$$\text{area ABO} = \int_{\rho=0}^5 \int_{\phi=0}^{\pi/2} \rho d\phi d\rho$$

$$= (\pi/2) \left( \frac{\rho^2}{2} \Big|_0^5 \right) = 6.25\pi$$



## Example 3.1 - solution

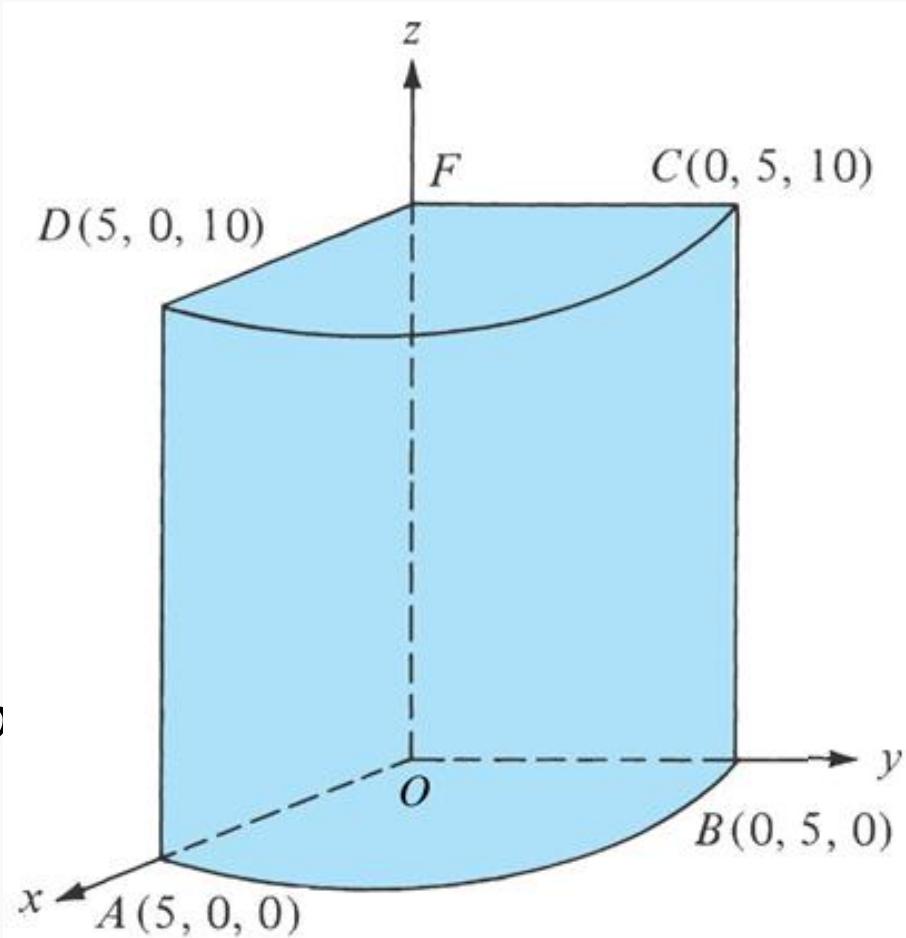
(e) for AOFD ,  $dS = d\rho dz$ ,  $\phi=0$

$$\text{area AOFD} = \int_{\rho=0}^5 \int_{z=0}^{10} d\rho dz = 50$$

(f) For volume ABCDFO,

$$dv = \rho d\phi dz d\rho$$

$$v = \int dv = \int_{\rho=0}^5 \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho d\phi dz d\rho$$
$$= 62.5\pi$$



# Line, Surface, and Volume Integrals

## (Line=Curve=Contour) Integral:

The Line integral  $\int_L \mathbf{A} \cdot d\mathbf{l}$  is integral of the tangential component of vector  $\mathbf{A}$  along  $L$ .

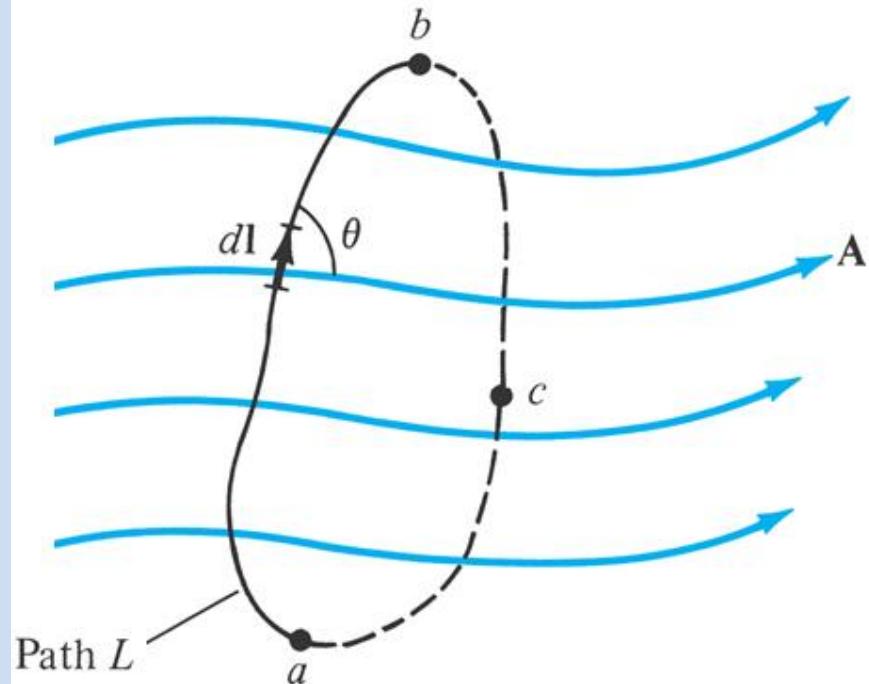
$$\bullet \int_L \mathbf{A} \cdot d\mathbf{l} = \int_a^b |\mathbf{A}| \cos \theta \, dl \rightarrow$$

**Line integral of A around L.**

- If the path of integration is

closed, such as abca,  $\oint_L \mathbf{A} \cdot d\mathbf{l} \rightarrow$

**Circulation of A along L.**



# Line, Surface, and Volume Integrals

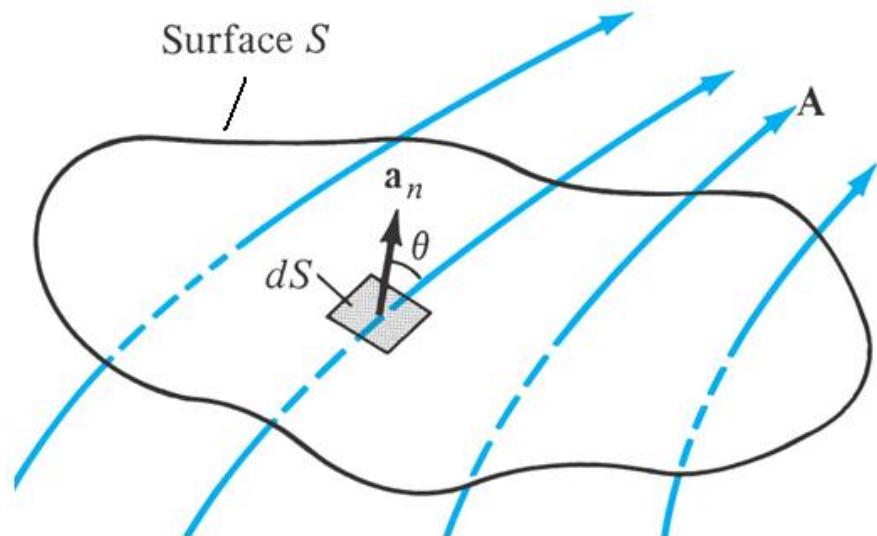
## Surface Integral:

Given vector  $\mathbf{A}$  continuous in a region containing the surface  $S \rightarrow$

The surface integral or the flux of  $\mathbf{A}$  through  $S$  is:

$$\Psi = \int_S \mathbf{A} \cdot d\vec{S}$$

$$\Psi = \int_S |\mathbf{A}| \cos \theta \, dS, \quad d\vec{S} = dS \, \mathbf{a}_n$$



Flux across  $dS$  is:  $d\Psi = |\mathbf{A}| \cos \theta \, dS = \mathbf{A} \cdot d\mathbf{S}$

→ Total Flux  $\Psi = \int d\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}$

# Line, Surface, and Volume Integrals

## Surface Integral:

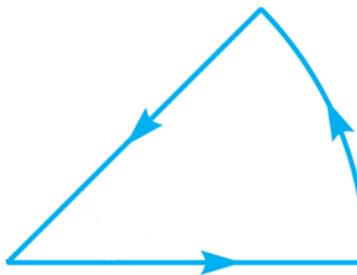
For a closed surface (defining a volume) :

$$\Psi = \oint_S \mathbf{A} \cdot d\vec{S}$$

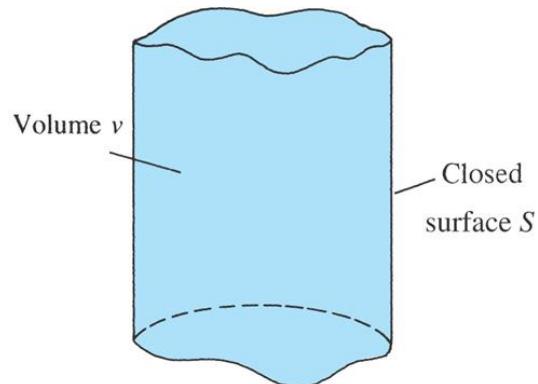
→ *The net outward flux of A from S*

### Notes:

A closed path defines an open surface.



A closed surface defines a volume.



# Line, Surface, and Volume Integrals

## Volume Integral:

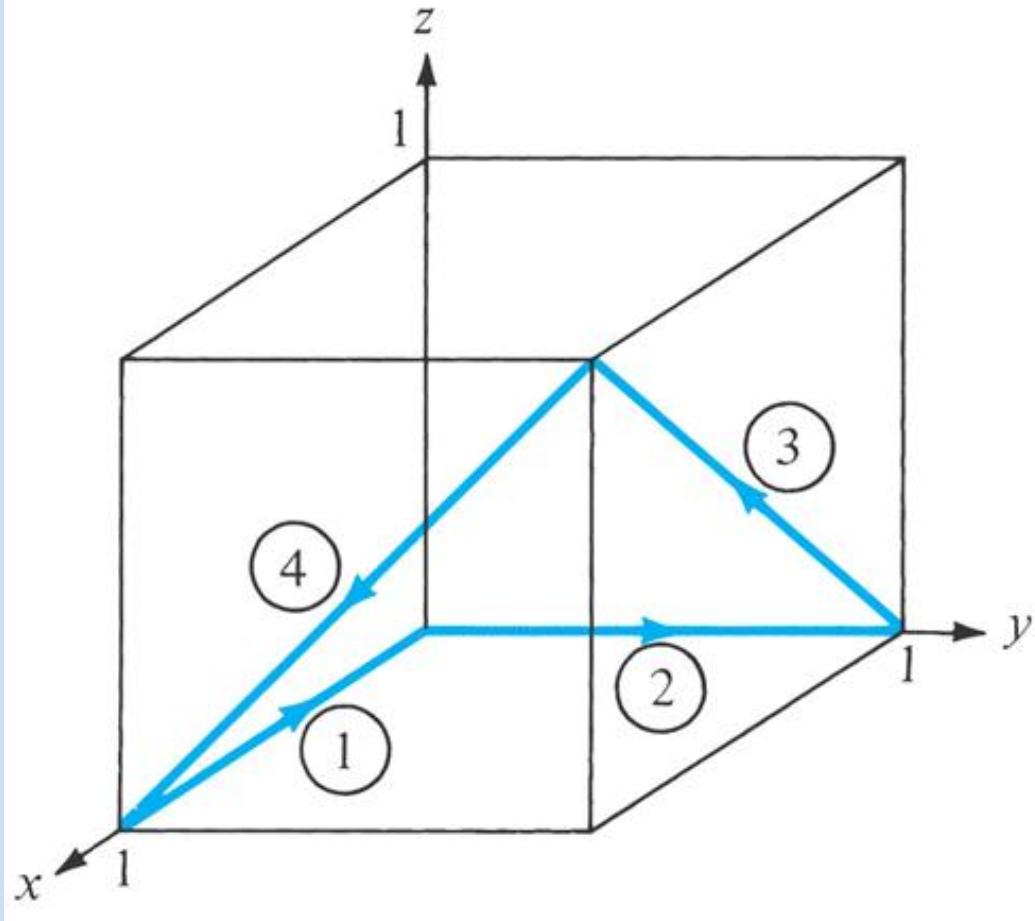
We define:

$$\int_v \rho_v \, dv$$

as the volume integral of the scalar  $\rho_v$  over the volume  $v$ .

## Example 3.2

Given that  $\mathbf{F} = x^2 \mathbf{a}_x - xz \mathbf{a}_y - y^2 \mathbf{a}_z$ . Calculate the circulation of  $\mathbf{F}$  around the (closed) path shown in the Figure.



## Example 3.2 - solution

The circulation of  $\mathbf{F}$  around  $L$  is:

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \left( \int_1 + \int_2 + \int_3 + \int_4 \right) \mathbf{F} \cdot d\mathbf{l}$$

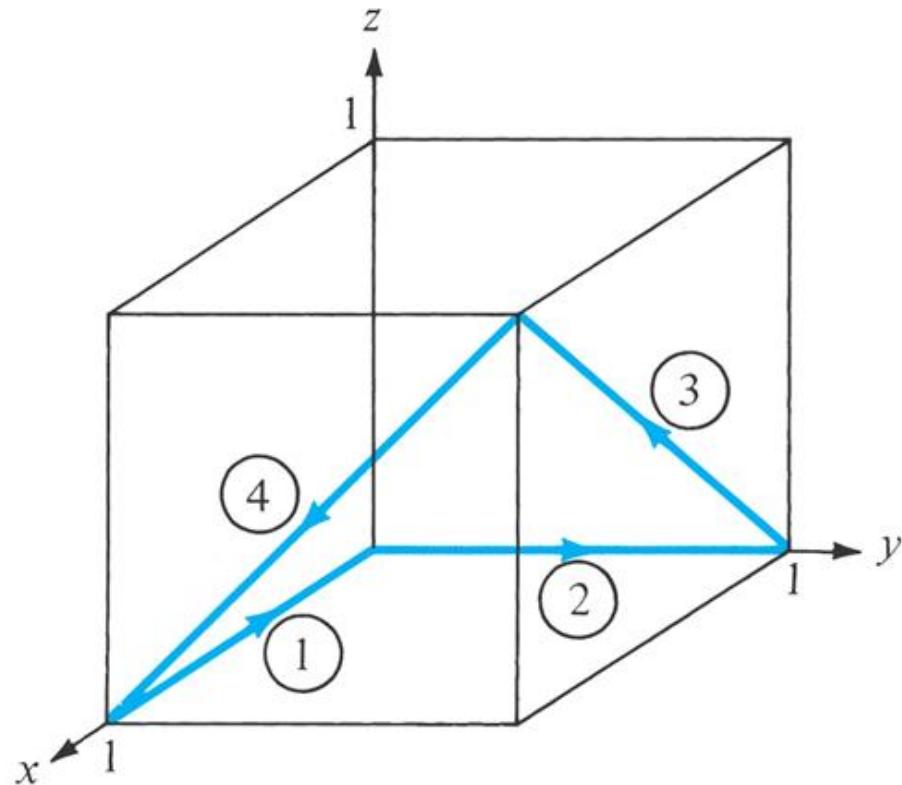
For segment 1,  $y=0, z=0$

$$\mathbf{F} = x^2 \mathbf{a}_x - xz \mathbf{a}_y - y^2 \mathbf{a}_z = x^2 \mathbf{a}_x$$

$$d\mathbf{l} = dx \mathbf{a}_x \text{ (+ve direction)}$$

$$\int_1^0 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 x^2 dx = -\frac{1}{3}$$

---



Segment 2,  $x = 0, z = 0, d\mathbf{l} = dy \mathbf{a}_y, \mathbf{F} \cdot d\mathbf{l} = 0 \rightarrow \int_2 \mathbf{F} \cdot d\mathbf{l} = 0$

## Example 3.2 - solution

Segment 3:  $y = 1$ ,  $\mathbf{F} = x^2 \mathbf{a}_x - xz \mathbf{a}_y - y^2 \mathbf{a}_z$ ,  $d\mathbf{l} = dx \mathbf{a}_x + dz \mathbf{a}_z$

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int x^2 dx - dz$$

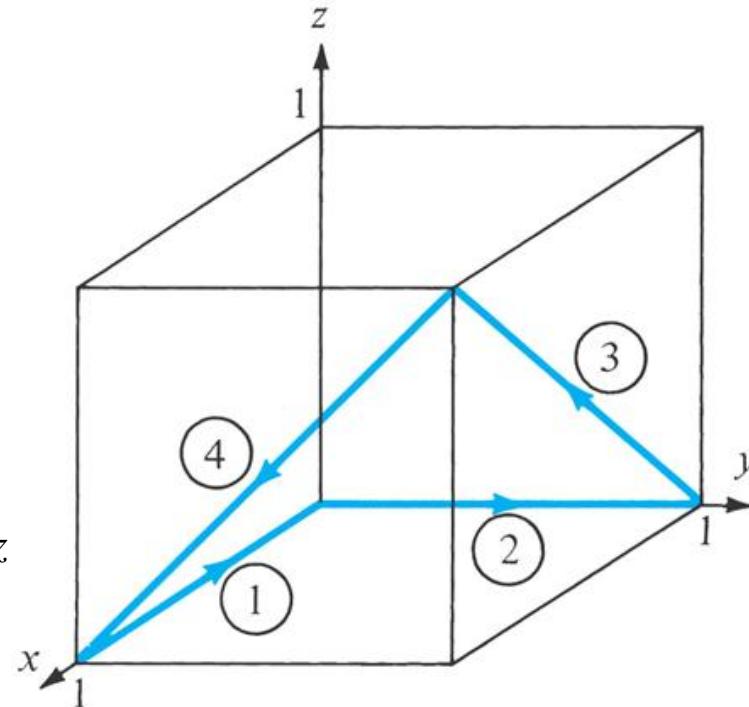
$$= \frac{x^3}{3} \Big|_0^1 - z \Big|_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

Segment 4:  $x = 1$ ,  $\mathbf{F} = \mathbf{a}_x - z \mathbf{a}_y - y^2 \mathbf{a}_z$

$$d\mathbf{l} = dy \mathbf{a}_y + dz \mathbf{a}_z$$

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int -z dy - y^2 dz, \text{ but on 4, } z = y, \quad dz = dy$$

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int (-y - y^2) dy = \frac{5}{6}, \quad \Rightarrow \oint_L \mathbf{F} \cdot d\mathbf{l} = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6} = -\frac{1}{6}$$



# Del Operator

The del operator, written as  $\nabla$ , is the vector differential operator.

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z$$

Useful in defining:

- (1) The gradient of a scalar  $V$ , written as  $\nabla V$
- (2) The divergence of a vector  $\mathbf{A}$ , written as  $\nabla \cdot \mathbf{A}$
- (3) The curl of a vector  $\mathbf{A}$ , written as  $\nabla \times \mathbf{A}$
- (4) The Laplacian of a scalar  $V$ ,  $\nabla^2 V$

# Del Operator

Del operator  $\nabla$  in Cylindrical Coordinates:  $(\rho, \phi, z)$

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x}$$

$$x = \rho \cos \phi, \quad y = \rho \sin \phi$$

*Cartesian*

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

→ use chain rule differentiation:

$$\frac{\partial}{\partial x} = \cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin \phi \frac{\partial}{\partial \rho} + \frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi}$$

# Del Operator

but  $a_x = \cos\phi a_\rho - \sin\phi a_\phi$ ,

$a_y = \sin\phi a_\rho + \cos\phi a_\phi$ ,  $a_z = a_z$

$\rightarrow \nabla$  in Cylindrical:

Cartesian

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

$$\nabla = \left( \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial}{\partial \phi} \right) (\cos\phi a_\rho - \sin\phi a_\phi)$$

$$+ \left( \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho} \frac{\partial}{\partial \phi} \right) (\sin\phi a_\rho + \cos\phi a_\phi) + \frac{\partial}{\partial z} a_z$$

$\Rightarrow$  In Cylindrical

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + a_z \frac{\partial}{\partial z}$$

$\Rightarrow$  In Spherical,

$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + a_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

# Del Operator

Cartesian

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

Cylindrical

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + a_z \frac{\partial}{\partial z}$$

Spherical,

$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + a_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

# Gradient of a Scalar

The gradient of a scalar  $V$  is a vector that represents both the magnitude and the direction of the maximum space rate of increase of  $V$

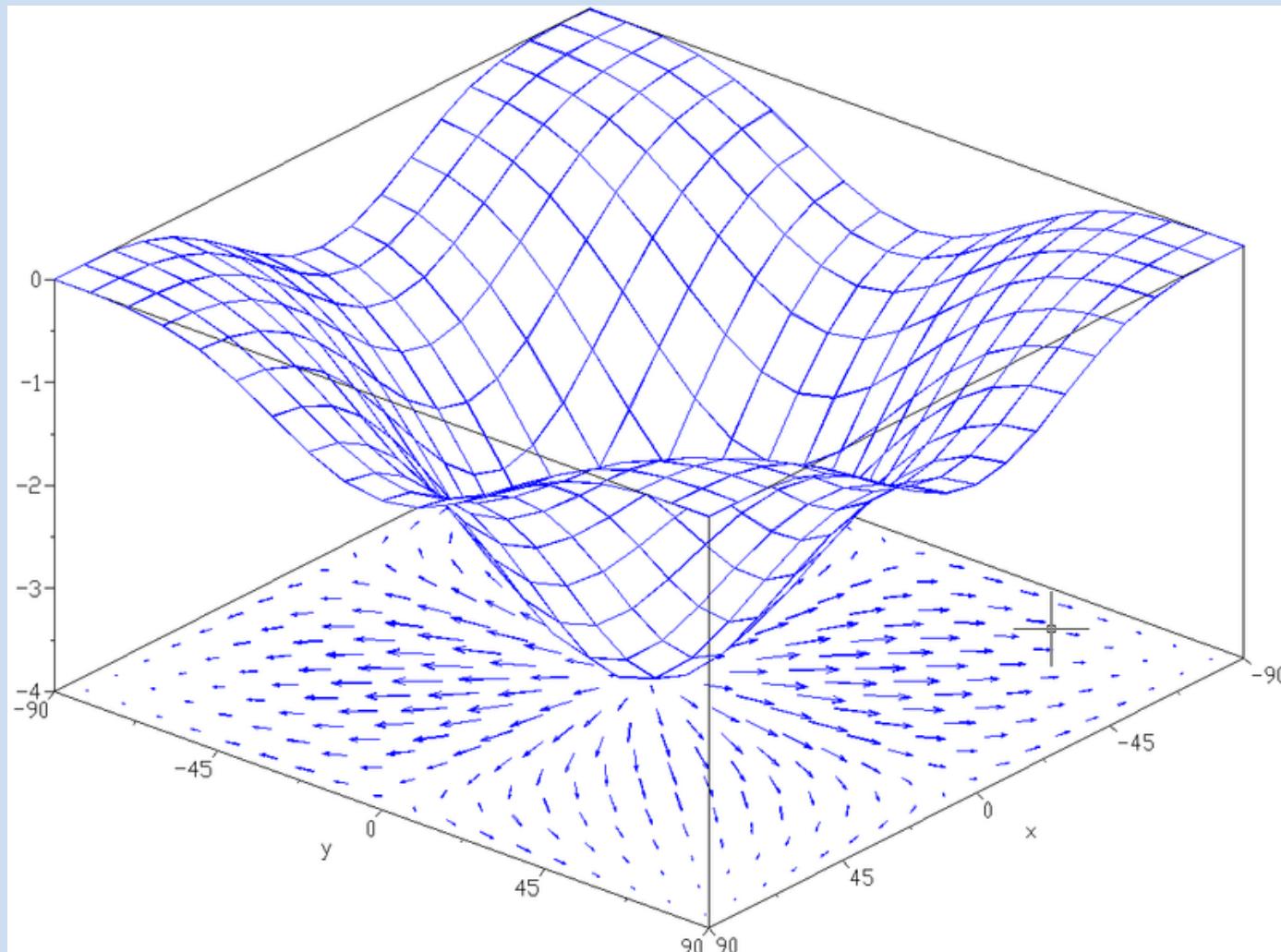
$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

- $|\nabla V|$  = maximum rate of change in  $V$
- $\nabla V$  points in the direction of the max. rate of change in  $V$

**Example:** Room with temperature given by a scalar field  $T(x,y,z)$ . At each point in the room, the gradient of  $T$  at that point will show the direction the temperature rises more quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

# Gradient of a Scalar

The gradient of the function  $f(x,y) = -(\cos^2x + \cos^2y)^2$  depicted as a projected vector field on the bottom plane



# Gradient of a Scalar

In Cartesian: →

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

For Cylindrical coordinates,

Recall

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + a_z \frac{\partial}{\partial z}$$

→

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$$

For Spherical coordinates,

Recall

$$\nabla = a_r \frac{\partial}{\partial r} + a_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + a_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

→

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} a_\phi$$

# Gradient of a Scalar

Notes: (a)  $\nabla(V + U) = \nabla V + \nabla U$

(b)  $\nabla(VU) = V\nabla U + U\nabla V$

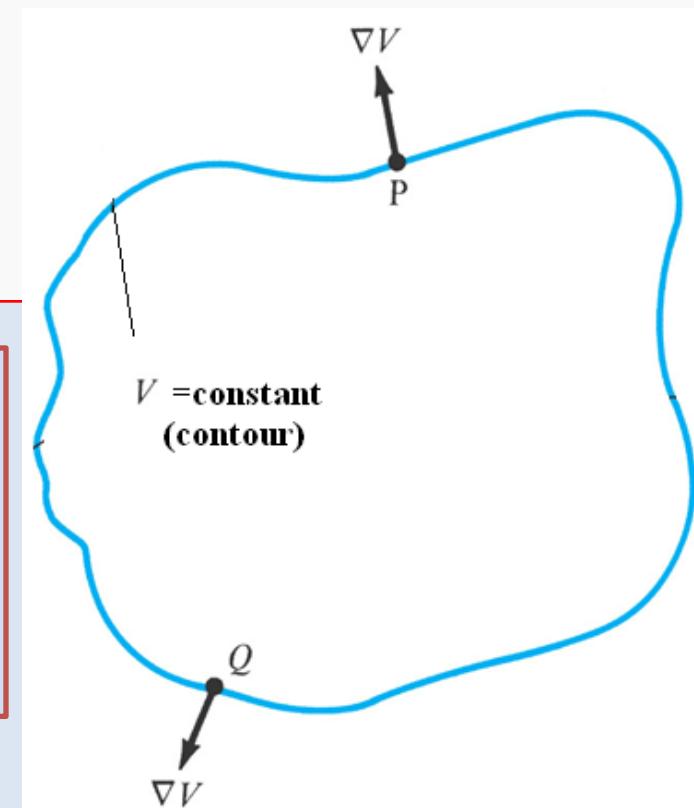
(c)  $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$

(d)  $\nabla V^n = nV^{n-1} \nabla V$

where  $U$  and  $V$  are scalars

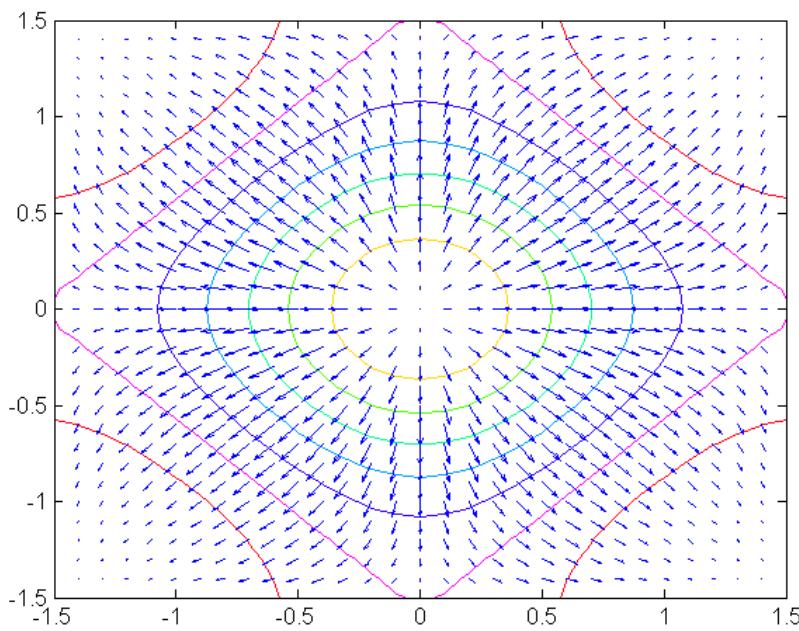
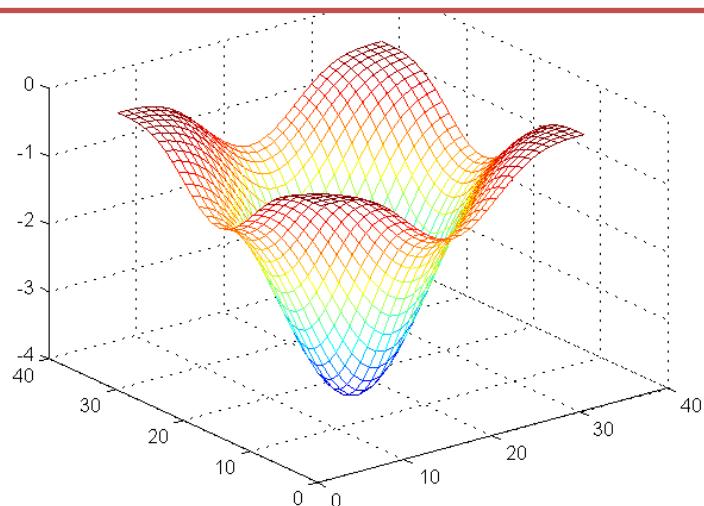
and  $n$  is an integer

The gradient at any point is  
perpendicular to the constant  $V$  surface  
that passes through that point.



$$f(x,y) = -(\cos^2 x + \cos^2 y)^2$$

$f(x,y) = \text{constant} \rightarrow \text{contour}$



$$f(x, y) = x^2 + y^2$$

$f(x, y) = C = \text{constant}$   
 $\rightarrow \text{Contour}$

---

$$f(x, y, z) = x^2 + y^2 + z^2$$

$f(x, y, z) = C$   
 $\rightarrow \text{level surface}$   
(ex: surface of sphere)

## Example 3.3

Find the gradient of the following scalar fields:

$$(a) V = e^{-z} \sin 2x \cosh y \quad (b) U = \rho^2 z \cos 2\phi$$

$$(c) W = 10r \sin^2 \theta \cos \phi$$

$$(a) \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

$$= 2e^{-z} \cos 2x \cosh y a_x + e^{-z} \sin 2x \sinh y a_y - e^{-z} \sin 2x \cosh y a_z$$

$$(b) \nabla U = \frac{\partial U}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial U}{\partial \phi} a_\phi + \frac{\partial U}{\partial z} a_z$$

$$= 2\rho z \cos 2\phi a_\rho - 2\rho z \sin 2\phi a_\phi + \rho^2 \cos 2\phi a_z$$

$$(c) \nabla W = \frac{\partial W}{\partial r} a_r + \frac{1}{r} \frac{\partial W}{\partial \theta} a_\theta + \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi} a_\phi$$

$$= 10 \sin^2 \theta \cos \phi a_r + 20 \sin \theta \cos \theta \cos \phi a_\theta - 10 \sin \theta \sin \phi a_\phi$$

## Example 3.5

Find the angle at which line  $x = y = 2z$  intersects the ellipsoid

$$x^2 + y^2 + 2z^2 = 10$$

To find the point of intersection:

$$x^2 + x^2 + 2(x/2)^2 = 10$$

$$\rightarrow 2x^2 + 0.5x^2 = 10 \rightarrow x = 2$$

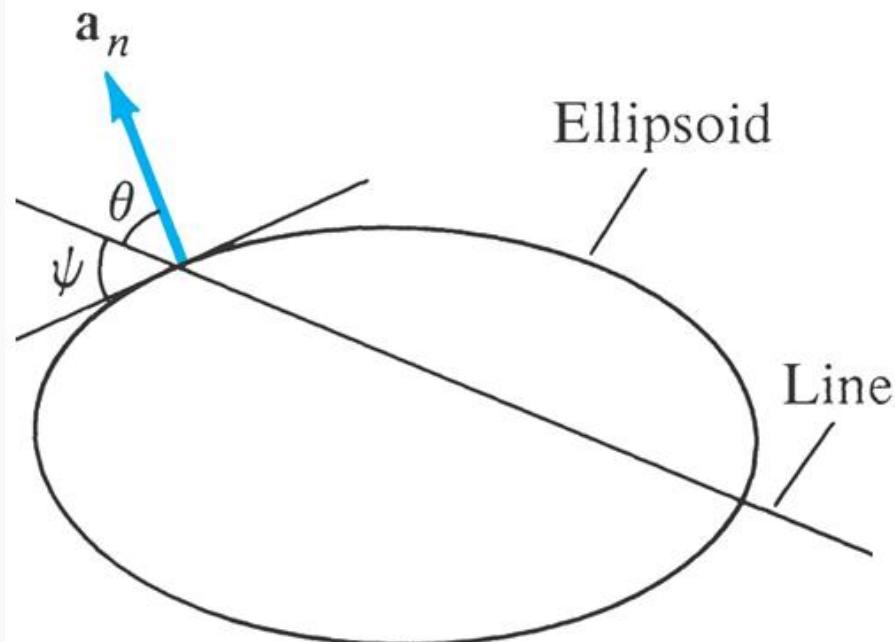
$$\rightarrow \text{point } (2,2,1)$$

$$f(x, y, z) = x^2 + y^2 + 2z^2 = 10$$

$$\rightarrow \text{surface of ellipsoid } [f(x, y, z) = \text{constant}]$$

$$\nabla f = 2x \mathbf{a}_x + 2y \mathbf{a}_y + 4z \mathbf{a}_z$$

$$\text{At } (2,2,1), \nabla f = 4 \mathbf{a}_x + 4 \mathbf{a}_y + 4 \mathbf{a}_z$$



## Example 3.5 - continued

Hence, a unit vector normal to the ellipsoid surface at P(2,2,1) is:

$$\mathbf{a}_n = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{4(a_x + a_y + a_z)}{\sqrt{48}}$$

$$= \pm \frac{(a_x + a_y + a_z)}{\sqrt{3}}$$

$$\cos \theta = \frac{\mathbf{a}_n \cdot \mathbf{r}}{|\mathbf{a}_n \cdot \mathbf{r}|}$$

$$= \frac{(1/\sqrt{3})(1,1,1) \cdot (2,2,1)}{\sqrt{9}} = \frac{5}{3\sqrt{3}}$$

$$\theta = 15.79^\circ$$

$$\rightarrow \psi = 90 - \theta = 74.2^\circ$$

