Vector Integration - GATE Study Material in PDF

In the previous article we have seen about the basics of vector calculus. In these GATE 2018 Study Notes we will learn about **Vector Integration**. A vector has both magnitude and direction whereas a scalar has only magnitude. Let us now see how to perform certain operations on vectors.

These GATE study material are useful for GATE EE, GATE EC, GATE CS, GATE ME, GATE CE and all other branches. Also useful for exams such as DRDO, IES, BARC, BSNL, ISRO etc. You can have these notes downloaded as PDF so that your exam preparation is made easy and you ace your paper. Before you get started, go through the basics of Engineering Mathematics.

Recommended Reading –

Types of Matrices

Properties of Matrices

Rank of a Matrix & Its Properties

Solution of a System of Linear Equations

Eigen Values & Eigen Vectors

Linear Algebra Revision Test 1

Laplace Transforms

[Limits, Continuity & Differentiability]

[Mean Value Theorems]

[Differentiation]

[Partial Differentiation]

[Maxima and Minima]

[Methods of Integration & Standard Integrals]

[Vector Calculus]

Rules of Vector Integration

Let $\frac{d}{dt}\left[\vec{f}(t)\right] = \vec{F}(t) \quad \Rightarrow \vec{f}(t) = \int \vec{F}(t)dt + c$ is called vector integration.

1. $\int_a^b \vec{f}(t)\, dt = F(b) - F(a)$

2. $\int_a^b \vec{f}(t)\, dt = -\int_b^a \vec{f}(t)\, dt$

3. $\int_a^b \vec{f}(t)\, dt = \int_a^c \vec{f}(t)\, dt + \int_c^b \vec{f}(t)\, dt$

Example 1:

Find the value of $\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot dt$

Solution:

$$\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot dt = \left[\frac{t^2}{2}\hat{i} + \frac{t^3}{3}\hat{j} + \frac{t^4}{4}\hat{k}\right]_0^1$$

$$= \left[\frac{1}{2}\right]\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{4}\hat{k}$$

Generally, integrals are classified as line integrals, surface integrals and volume integrals.

Line Integrals

An integral which is to evaluated along the curve is called line integral

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Consider $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $d\vec{r} = \hat{i}dx + dy\hat{j} + dz\hat{k}$

$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_c [F_1\hat{i} + F_2\hat{j} + F_3\hat{k}] \cdot [\hat{i}dx + \hat{j}dy + \hat{k}dz]$

$= \int_c [F_1 dx + F_2 dy + F_3 dz]$ where C is open curve

If C is a close curve then $\oint \vec{F} \cdot d\vec{r}$ is called the circulation then we need to Put $x = r\cos\theta$, $y = r\sin\theta$ and $\theta = 0$ to $\theta = 2\pi$ solve the given integration.

Example 2:

Find the value of $\int_c \vec{F} \cdot d\vec{r}$ if $F = x\hat{i} - y\hat{j}$ and c is given by $x^2 + y^2 = 4$

Solution:

$\int_c \vec{F} \cdot d\vec{r} = \int_c F_1 dx\hat{i} + F_2 dy\hat{j}$

$= \int_c (xdx - ydy)$

Put, $x = 2\sin\theta\,, y = 2\cos\theta$

$dx = 2\cos\theta d\theta$ and $dy = -2\sin\theta d\theta$

$\int_0^{2\pi} [2\sin\theta(2\cos\theta)d\theta] - [2\cos\theta(-2\sin\theta)]\, d\theta$

$\int_0^{2\pi} 8\cos\theta\sin\theta\, d\theta = 4\int_0^{2\pi} \sin 2\theta\, d\theta$

$= 4 \cdot \frac{-1}{2}[\cos 2\theta]_0^{2\pi} = -2[1-1] = 0$

$\therefore \int_c \vec{F} \cdot d\vec{r} = 0$

Note:

If $\vec{F}$ is force then the total work done by a force is $\int_c \vec{F} \cdot d\vec{r}$.

If $\int_c \vec{F} \cdot d\vec{r} = 0 \Rightarrow \vec{F}$ is called conservative force.

Green's Theorem

If M(x,y) and N (x,y) are continuous and having continuous first order partial derivatives bounded by a closed curve 'C' then

$$\int_c (Mdx + Ndy) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx$$

Example 3:

Find the value of $\int_c (3x + 4y)\, dx + (2x - 2y)dy$ where $C: \; x^2 + y^2 = 4$

Solution:

Here $M = 3x + 4y, \;\; N = 2x - 3y$

$\frac{\partial M}{\partial y} = 4 \qquad \frac{\partial N}{\partial x} = 2$

By Green's theorem $\int_R (Mdx + Ndy) = \int\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$= -2 \int\int dxdy = (-2)(\pi r^2) = -8\pi$

Multiple Integrals

They are mainly classified as

1. Double Integrals
2. Triple Integrals

1. Double Integrals

$\int_R \int f(x,y)\, dx\, dy$ is called the double integral where $R \in [(x_1, x_2)(y_1, y_2)]$

Methods to Evaluate Double Integrals

Method 1:

If y_1, y_2 are functions of x only and x_1, x_2 are constants then the order of integration is first integral is with respect to "y" treating x- as a constant then the remaining expression integrate with respect to x.

$$\int_R \int f(x,y)dydx = \int_{x_1=a}^{x_2=b} \int_{y_1=f(x_1)}^{y_2=f(x_2)} f(x,y) \cdot dy \cdot dx$$

Example 4:

Find the value of $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Solution:

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \left[\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{(\sqrt{1+x^2})}{\sqrt{1+x^2}}\right] \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) \cdot dx = \frac{\pi}{4} \cdot [\sin h^{-1}(x)]_0^1$$

$$= \frac{\pi}{4} \cdot [\sin h^{-1}(1) - \sin h^{-1}(0)]$$

$= \frac{\pi}{4} \cdot \left[\log\left(1+\sqrt{1+1}\right)\right] = \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$

Method 2:

If x_1, x_2 are functions of y only, and y_1, y_2 are constant then the order of integration is first w.r.t x treating y as constant and then integrate remaining expression w.r.t. y

$$\int_R \int f(x,y)dxdy = \int_{y_1=c}^{y_2=d} \int_{x_1=\phi(y)}^{x_2=\phi(y)} f(x,y) \cdot dxdy$$

Example 5:

Find the value of $\int_{y=0}^{1} \int_{x=\sqrt{y}}^{y} (x+y)\, dxdy$

Solution:

$$\int_0^1 \left[\frac{x^2}{2} + xy\right]_{\sqrt{y}}^{y} \cdot dy = \int_0^1 \left(\frac{y^2}{2} + y^2\right) - \left(\frac{y}{2} + y^{\frac{3}{2}}\right)$$

$$= \left[\frac{y^3}{6} + \frac{y^3}{3} - \frac{y^2}{4} - \frac{2y^{\frac{5}{2}}}{5}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{4} - \frac{2}{5}$$

$$= \frac{10+20-15-24}{60} = \frac{30-39}{60} = \frac{-9}{60} = \frac{-3}{20}$$

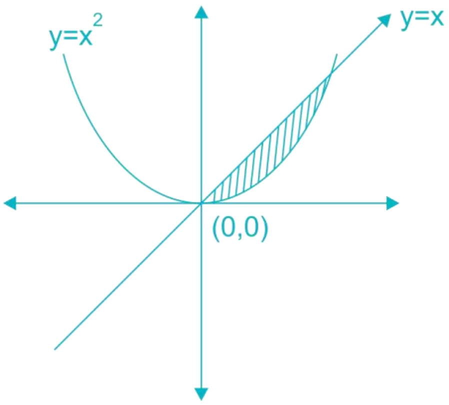
Area Using Double Integral:

$$\text{Area} = \int\int dA = \int\int_R dydx = \int_{x_1}^{x_2}\int_{y_1}^{y_2} dydx$$

Example 6:

Find the area bounded between the curves $y = x^2$, $y = x$

Solution:



Area $= \int_0^1 \int_{x^2}^{x} \cdot dy.dx$

$= \int_0^1 [y]_{x^2}^{x} \cdot dx = \int_0^1 (x - x^2) \cdot dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

2. Triple Integrals

Triple integral is defined as $\int \int_R \int f(x, y, z)dxdydz$

The order of integration:-

If z_1, z_2 are function of x and y and y_1, y_2 are function of x and x_1, x_2 are constant the order of integration is

$$\int_{x_1=a}^{x_2=b} \int_{y_1=f(x)}^{y_2=f(x)} \int_{z_1=\phi(x,y)}^{z_2=\psi(x,y)} \left[\left[\left[f(x,y,z).\underset{(1)}{\underset{\downarrow}{dz}}\right] \underset{(2)}{\underset{\downarrow}{dy}}\right] \underset{(3)}{\underset{\downarrow}{dx}}\right]$$

Example 7:

Find the value of $\int_0^1 \int_0^x \int_0^y xyz\ dz\ dy\ dx$

Solution:

$\int_0^1 \int_0^x \int_0^y xyz\, dz\, dy\, dx = \int_0^1 \int_0^x xy \left[\frac{z^2}{2}\right]_0^y dy \cdot dx = \frac{1}{2}\int_0^1 \int_0^x xy^3 \cdot dy \cdot dx$

$= \frac{1}{2}\left[\int_0^1 x \frac{[y^4]_0^x}{4} \cdot dx\right]$

$= \frac{1}{8}\left[\int_0^1 x^5 \cdot dx\right] = \frac{[x^6]_0^1}{8(6)} = \frac{1}{48}$

Surface Integrals

An integral which is to be evaluated over a surface is called surface integral.

Mathematical formula for surface integral is $= \int_s \int \vec{F} \cdot \vec{N}\, ds$

where $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

$\vec{N} =$ Outward unit normal vector and ds $=$ projection of surface on to the planes

Method To Evaluate Surface Integral:

1. If the surface S is on to XY (Z = 0) plane then

$$\int \int_S \vec{F} \cdot \hat{N}\, ds = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \vec{F} \cdot \hat{N} \frac{dydx}{|\hat{N}\cdot\hat{k}|}$$

where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{k}$.

2. If the surface S is on to XZ (Y = 0) plane then

$$\int \int_S \vec{F} \cdot \hat{N}\, ds = \int_{x_1}^{x_2} \int_{z_1}^{z_2} \vec{F} \cdot \hat{N} \frac{dzdx}{|\hat{N}\cdot\hat{\jmath}\,|}$$

where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{\jmath}$

3. If the surface S is on to YZ (X = 0) plane then

$$\int_S \int \vec{F} \cdot \hat{N}\, ds = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \vec{F} \cdot \hat{N} \cdot \frac{dydz}{|\hat{N} \cdot \hat{\imath}|}$$

where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{\imath}$

Example 8:

The value of $\int \int_S \vec{F} \cdot \hat{N}\, ds$ where $\vec{F} = z\hat{\imath} + x\hat{\jmath} - 3y^2\hat{k}$ and s is the surface of cylinder

$x^2 + y^2 = 16$ included in the first octant between $z = 0, z = 5$ is.

Solution:

$\phi = x^2 + y^2 - 16$ and

$$\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{\imath}(2x) + \hat{\jmath}(2y)}{2\sqrt{x^2+y^2}} = \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}} = \frac{x\hat{\imath} + y\hat{\jmath}}{4}$$

$$\vec{F} \cdot \hat{N} = \frac{zx + xy}{4}$$

$$\int_s \int \vec{F} \cdot \hat{N}\, ds = \int \int_S \frac{x(z+y)}{4} \cdot ds$$

$x^2 + y^2 = 16$ put $x = 0 \Rightarrow y = \pm 4$

Let the surface s is projected on to YZ plane, and $|\hat{N} \cdot \hat{\imath}| = \frac{x}{4}$

$$\int \int_S \vec{F} \cdot \hat{N}\, ds = \int_{z=0}^{5} \int_0^4 \frac{x}{4}(z + y) \cdot \frac{dzdy}{x}(4)$$

$$\int_{z=0}^{5} \left(zy + \frac{y^2}{2}\right) \cdot dz = \int_0^5 (4z + 8)dz = [2z^2 + 8z]_0^5$$

$= 50 + 40 = 90$

Stoke's Theorem

According to this theorem, let S be the two sided open surface bounded by a closed curve "C" and $\vec{F}$ be differentiable vector function then

$$\oint_C \vec{F} \cdot d\vec{r} = \int \int_S (\vec{\nabla} \times \vec{F}) \cdot \hat{N} ds$$

It gives relation between line integral and surface integral.

Example 9:

By Stoke's theorem the value of $\int r \cdot d\vec{r}$ where c is $x^2 + y^2 = 16, z = 0$ is where

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Solution:

$$\int \vec{r} \cdot d\vec{r} = \int \int (\vec{\nabla} \times \vec{r}) \cdot \hat{N} \cdot ds$$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ x & y & z \end{vmatrix} = \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(0) = 0$$

$$\int_c \vec{r} \cdot dr = \int \int (0) \cdot \hat{N}\,(ds) = 0$$

Volume Integrals

The integral $\int \int \int_V \vec{F} dv$ is called volume integral and it is given as

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}) \cdot dzdydx$$

$$= \hat{\imath} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_1 dzdydx + \hat{\jmath} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_2 dzdydx + \hat{k} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_3 dzdydx$$

Gauss Divergence Theorem

It gives relation between surface integral to volume integral.

Definition:

Let 'V' be the volume bounded by a closed surface S and $\vec{F}$ be a differentiable vector function then

$$\int \int_S (\vec{F} \cdot \hat{N}) ds = \int \int_V \int (\vec{\nabla} \cdot \vec{F})\, dv$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dzdydx$$

Example 10:

By Gauss Divergence theorem find the value of $\int \int_s \vec{r} \cdot \hat{N} ds$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$

Solution:

$$\vec{\nabla} \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\int \int_S \vec{r} \cdot \hat{N} ds = \int \int \int (3) \cdot dv$$

$$= 3\left(\frac{4\pi}{3} r^3\right) = 4\pi$$

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