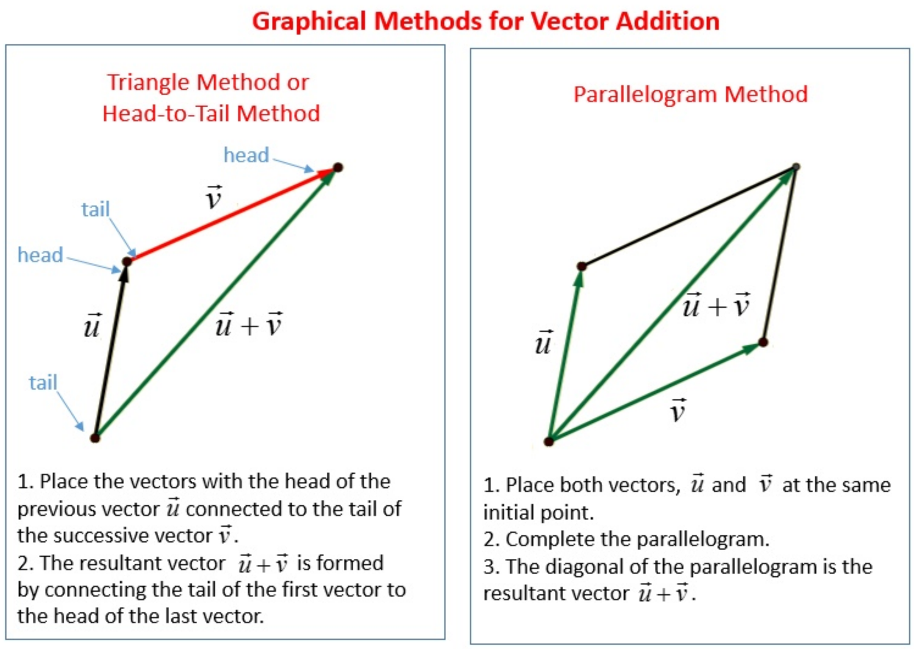
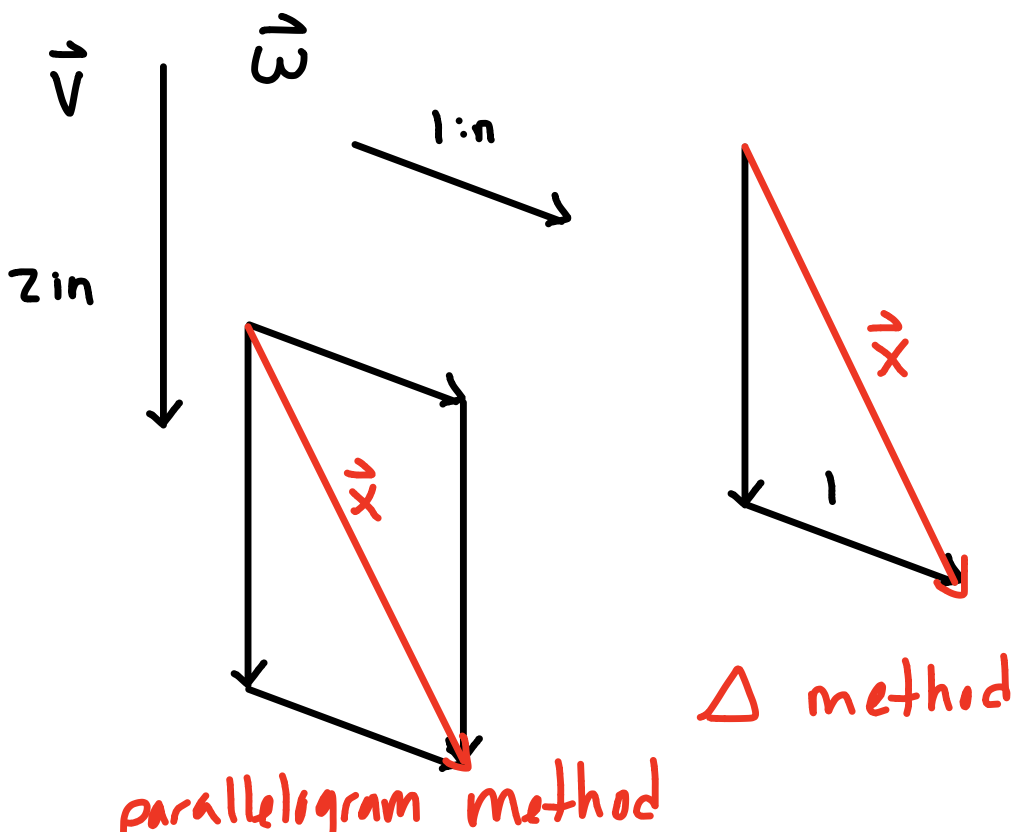
Vectors

- A vector is a quantity that has both magnitude and direction.
- A vector is represented by a directed line segment.
- Magnitude is the length of the line segment.
- If a vector has its initial point (starting point) at the origin, it is in standard position.
- The direction of the vector is the directed angle between the positive x-axis and the vector.
- Two vectors are equal if they have the same direction and magnitude.
- Two vectors are opposites if they have the same magnitude and opposite directions.
- The sum of two vectors is called the resultant.

Geometric Vectors



Ex.1 Find the sum of v and w.



Algebraic Vectors

Let $P_1(x_1, y_1)$ be the initial point of a vector and $P_2(x_2, y_2)$ be the terminal point.

Component Form $\quad \overrightarrow{P_1P_2}$ is $\langle x_2 - x_1, y_2 - y_1 \rangle$

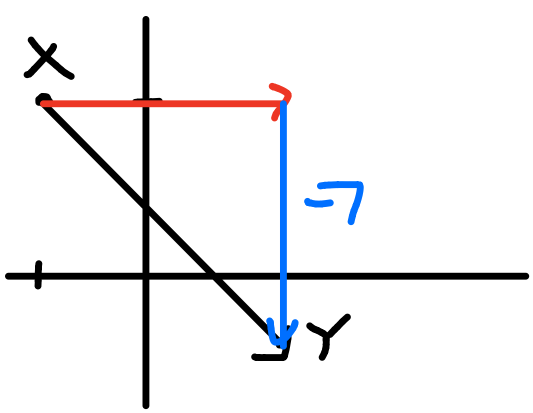
Any vector can be expressed in i j form

$$\vec{a} = \langle a_1 a_2 \rangle = a_1 i + a_2 j$$

The magnitude of a vector can be found by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Ex.2 Write the ordered pair that represents the vector from X(-3,5) to Y(4,-2). Then find the magnitude of $\overrightarrow{XY}$.



$\langle x_2 - x_1, y_2 - y_1 \rangle$

$\langle 7, -7 \rangle$

$7i - 7j$

$a^2 + b^2 = c^2$

$7^2 + (-7)^2 = c^2$

$\sqrt{49 + 49} = \sqrt{c^2}$

Vector operations

$\vec{a}\langle a_1, a_2\rangle \quad \vec{b}\langle b_1, b_2\rangle$

$\sqrt{98} = ($

$\boxed{7\sqrt{2} = |\overrightarrow{XY}|}$

Addition: $\vec{a} + \vec{b} = \langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1+b_1, a_2+b_2\rangle$

Subtraction: $\vec{a} - \vec{b} = \langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1-b_1, a_2-b_2\rangle$

Scalar Multiplication: $K\vec{a} = K\langle a_1, a_2\rangle = \langle Ka_1, Ka_2\rangle$

Ex.3 Let $\vec{m} = \langle 5,-7\rangle$, $\vec{n} = \langle 0,4\rangle$, and $\vec{p} = \langle -1,3\rangle$

A) m + p

$\langle 5,-7\rangle + \langle -1,3\rangle$

$\langle 5-1, -7+3\rangle$

$\boxed{\langle 4,-4\rangle}$

B) m - n

$\langle 5,-7\rangle - \langle 0,4\rangle$

$\boxed{\langle 5,-11\rangle}$

C) 7p

$7\langle -1,3\rangle$

$\boxed{\langle -7,21\rangle}$

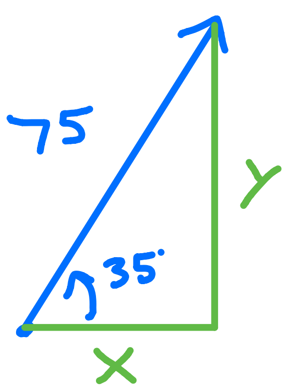
D) 2m + 3n -p

$2\langle 5,-7\rangle + 3\langle 0,4\rangle - \langle -1,3\rangle$

$\langle 10,-14\rangle + \langle 0,12\rangle + \langle 1,-3\rangle$

$$\boxed{\langle 11,-5\rangle}$$

Ex.4 A ship leaving port sails for 75 miles in a direction 35 degrees north of due east. Find the magnitude of vertical and horizontal components.



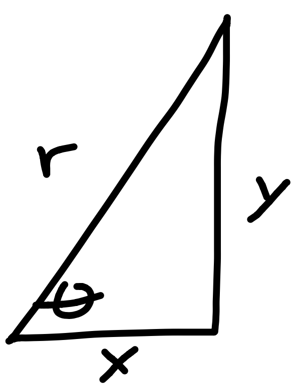
$$75(\cos 35) = \left(\frac{x}{\cancel{75}}\right)\cancel{75}$$

$$x = 61.4$$

$$75(\sin 35) = \left(\frac{y}{\cancel{75}}\right)\cancel{75}$$

$$y = 43$$

$$\boxed{\langle 61, 43\rangle}$$



$$r(\cos\theta) = \left(\frac{x}{\cancel{r}}\right)\cancel{r}$$

$$\boxed{x = r(\cos\theta)}$$

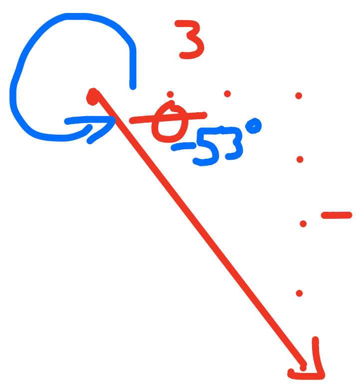
$$r(\sin\theta) = \left(\frac{y}{\cancel{r}}\right)\cancel{r}$$

$$\boxed{y = r\cdot\sin\theta}$$

Ex.5 Find the direction of the vector.

A) $\langle 3, -4 \rangle$

$\boxed{307^\circ}$

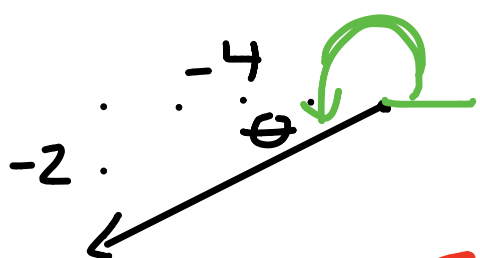


$\tan\theta = \frac{-4}{3}$

$\theta = \tan^{-1}\left(\frac{-4}{3}\right)$

$\theta = -53^\circ$

B) $\langle -4, -2 \rangle$



$\theta = \tan^{-1}\left(\frac{-2}{-4}\right)$

$\theta = 27^\circ$

$27 + 180 = \boxed{207^\circ}$

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

$r = \sqrt{(x^2 + y^2)}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Dot Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

- $\vec{a}$ and $\vec{b}$ are perpendicular, if their dot product is zero. (Orthogonal)
- The dot product can be used to find the angle between two vectors.

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|a||b|}\right)$$

Ex.1 Find the dot product and determine if the vectors are parallel, ~~orthogonal~~, or neither.

A) $\vec{a} = \langle 1, -4 \rangle \qquad \vec{b} = \langle 2, 2 \rangle$

$$(1)(2) + (-4)(2)$$
$$2 - 8 = \boxed{-6}$$

neither

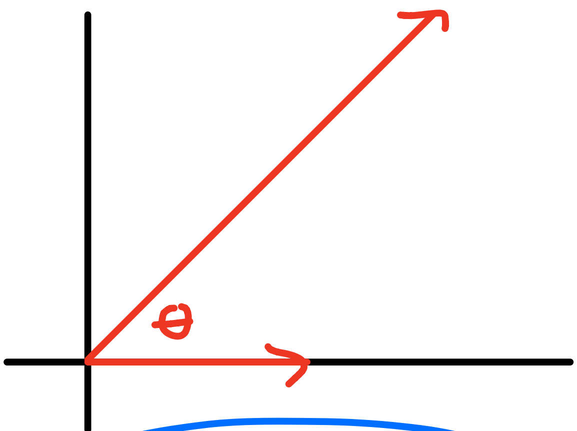
B) $\vec{a} = \langle 1, -4 \rangle \qquad \vec{c} = \langle 4, 1 \rangle$

$$(1)(4) + (-4)(1)$$
$$4 - 4$$
$$0$$

orthogonal

Ex.2 Find the angle between u and v.

$\vec{u} = \langle 3,0 \rangle \qquad \vec{v} = \langle 5,5 \rangle$



$\vec{u} \cdot \vec{v} = 15$

$|\vec{u}| = 3$

$|\vec{v}| = 50$

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|a||b|}\right)$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{9}\sqrt{50}}\right)$$

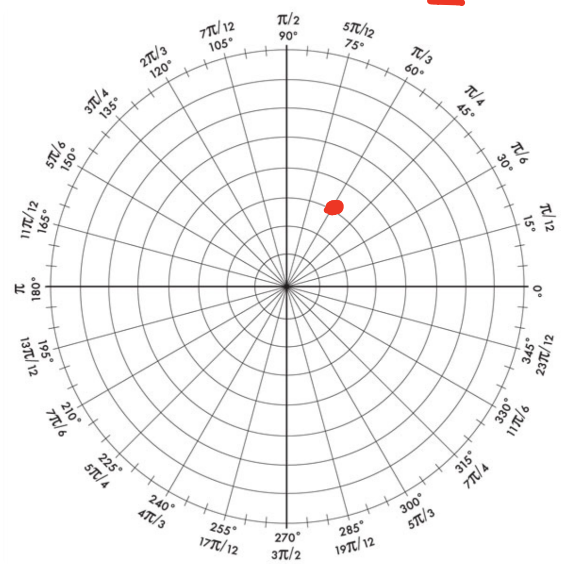
$$\boxed{\theta = 45^\circ}$$

$m_u = \sqrt{x^2 + y^2}$

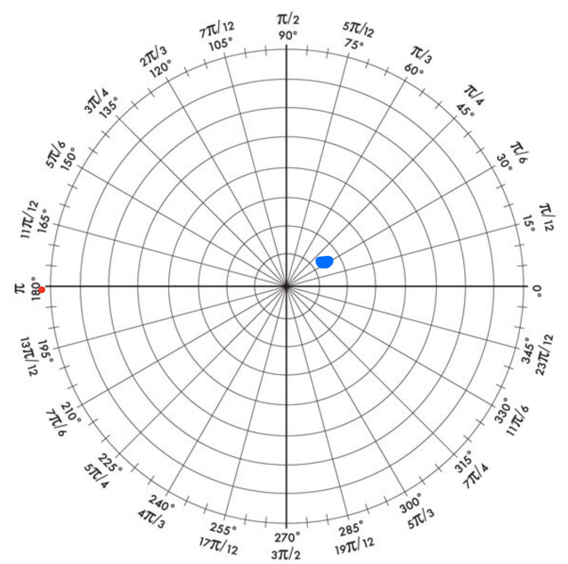
Polar Coordinates (r, θ)

Ex.1 Graph $P(3, 60^\circ)$

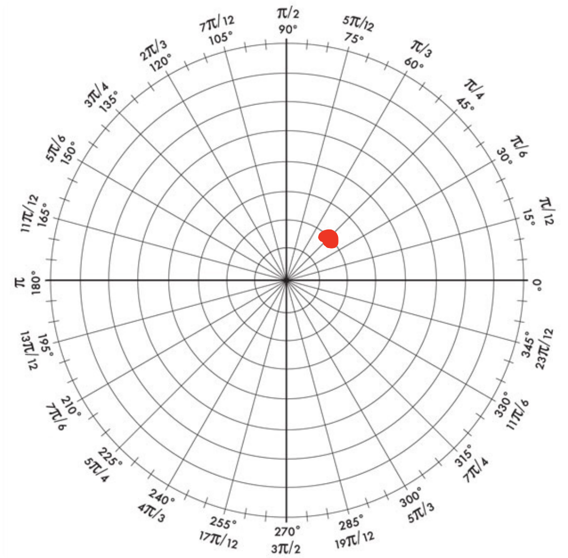
r = radius
θ = angle



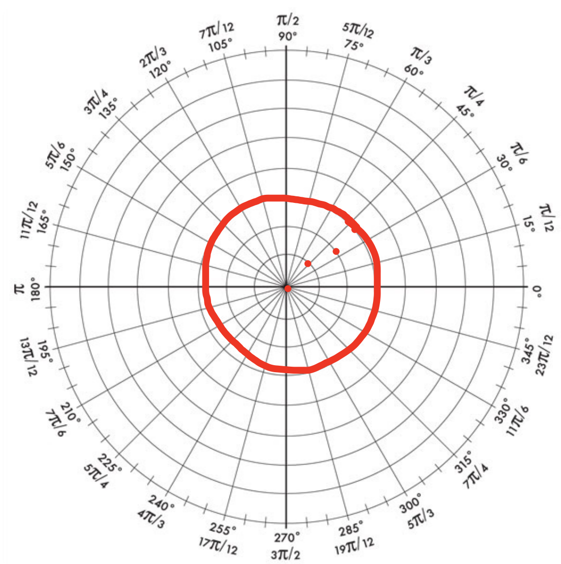
Ex.2 $Q\left(-1.5, \frac{7\pi}{6}\right)$



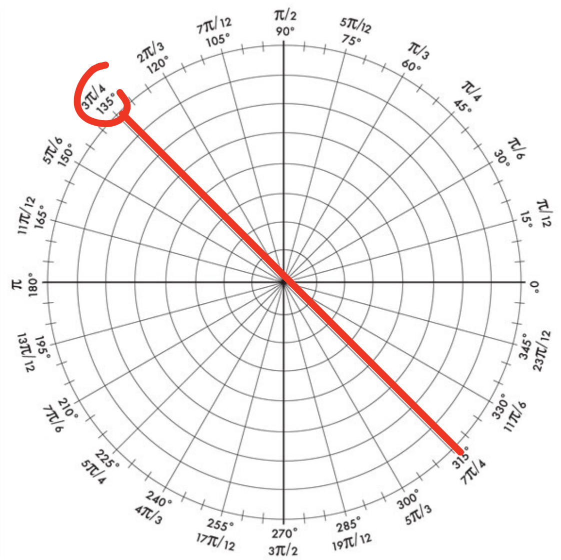
Ex.3 $R(-2, -135^\circ)$



Ex.4 $r=3$



Ex.5 $\theta = \frac{3\pi}{4}$



Distance Formula in Polar Plane

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Ex.6 Find the distance between

$L_1(450, 30^\circ)$ and $L_2(600, -50^\circ)$

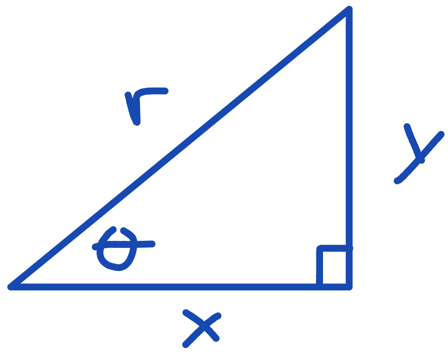
$r_1 \quad \theta_1 \qquad r_2 \quad \theta_2$

$$d = \sqrt{(450)^2 + (600)^2 - 2(450)(600)\cos(-50 - 30)}$$

$$\boxed{d = 685}$$

Converting from Polar to Cartesian

- Polar Coordinates (r, θ)
- Cartesian Coordinates (x, y)

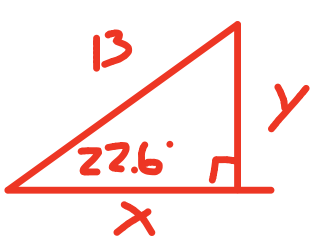


$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

$$(r \cdot \cos\theta, \ r \cdot \sin\theta)$$

Ex.7 convert (13, 22.6°) to Cartesian Coordinates.



$$(13 \cdot \cos 22.6^\circ, \ 13 \cdot \sin 22.6^\circ)$$

$$\boxed{(12, 5)}$$

Convert from Cartesian to Polar

$$r=\sqrt{x^2+y^2}$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

Ex.8 (-3,10)

(r,θ)

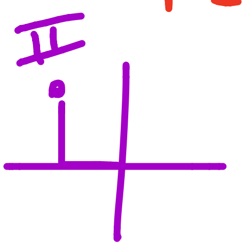
$$r=\sqrt{(-3)^2+(10)^2}$$

$$r=\sqrt{109}\approx 10.4$$

$$\theta=\tan^{-1}\left(\frac{10}{-3}\right)$$

$$\theta=-73^\circ+180^\circ$$

$$\theta=107^\circ$$



$$\boxed{(10, 107^\circ)}$$

Ex.9 (5,-8)

(r,θ)

$$r=\sqrt{5^2+(-8)^2}$$

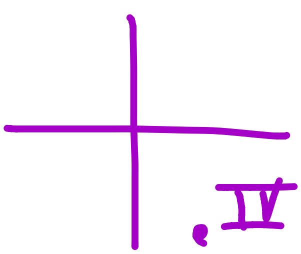
$$r=\sqrt{89}\approx 9.4$$

$$\theta=\tan^{-1}\left(\frac{-8}{5}\right)$$

$$\theta=-58^\circ$$

+180

+180



$$\boxed{(9, 302^\circ) \text{ or } (9, -58^\circ)}$$

Rewriting Equations

Write the Polar Equation to rectangular form.

- **Use substitution to replace r and theta with x and y.**
- **Substitute in $r = \sqrt{x^2 + y^2}$, $x = r\cos(\theta)$, or $y = r\sin(\theta)$.**

$\tan\theta = \frac{y}{x}$

Ex.1 $r(r) = (6\cos\theta)r$

$(r)^2 = 6(r\cos\theta)$

$\sqrt{x^2+y^2}^{\,2} = 6x$

$\boxed{x^2+y^2 = 6x}$

Rectangular equations to Polar form.

Ex.2 $(x-3)^2 + y^2 = 9$

$(r\cos\theta - 3)^2 + (r\sin\theta)^2 = 9$

$(r\cos\theta - 3)(r\cos\theta - 3) + r^2\sin^2\theta = 9$

$r^2\cos^2\theta - 3r\cos\theta - 3r\cos\theta + \cancel{9} + r^2\sin^2\theta = \cancel{9}$

$-9 \qquad -9$

$r^2\cos^2\theta + r^2\sin^2\theta - 6r\cos\theta = 0$

$r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) - 6r\cos\theta = 0$

$r^2 - 6r\cos\theta = 0$

Vector Direction and Magnitude Word Problems

- Rewrite the vector to component form.

$\langle r\cos\theta, r\cdot\sin\theta \rangle$

- Find resultant vector by summing the two vectors
- Find the magnitude.

$\text{magnitude} = \sqrt{x^2+y^2}$

- Find the direction.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Ex.1 To reach the final destination a pilot is plotting a course that will result in a velocity of 450 mph in a direction N 60 W. There is a wind blowing at 50 mph from due south. Find the direction and the speed of the plane.

$\langle 450\cdot\cos 150, 450\sin 150 \rangle$

$\langle -390, 225 \rangle$

$\langle 50\cdot\cos(90), 50\sin(90) \rangle$

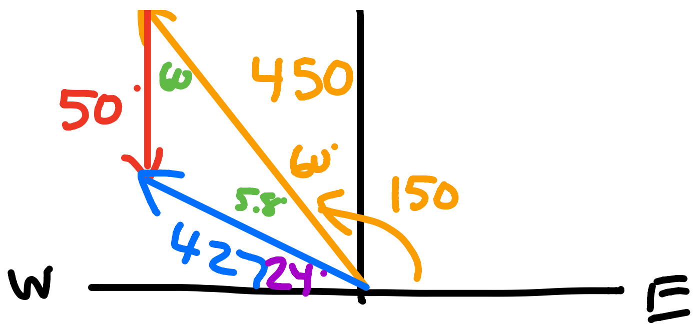
$\langle 0, 50 \rangle$

$\langle -390, 175 \rangle$

$m = 427$

$\theta = -24°$

N



$180 - 24 = \boxed{156°}$

Ex.2 what is the magnitude and direction of the resultant of a 105 Newton force along the positive x axis and a 110 Newton ford at an angle of 25.

① $\langle r\cdot\cos\theta, r\cdot\sin\theta \rangle$

$\langle 105\cos 0, 105\cdot\sin 0 \rangle$ $\langle 110\cos 25, 110\cdot\sin 25 \rangle$

$\langle 105, 0 \rangle$ $\langle 100, 46 \rangle$

sum: $\langle 205, 46 \rangle$

② magnitude $= \sqrt{x^2+y^2}$

$= \sqrt{205^2+46^2}$

$= 210$

③ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}\left(\frac{46}{205}\right)$

$= 13°$

} direction

Ex.3 Find the magnitude and direction of two forces of 250 pounds and 45 pounds at angles of 25 and 250 with the x-axis respectively.

$\langle 250\cos 25, 250\sin 25\rangle$

$\langle 227, 106\rangle$

$\langle 45\cdot\cos 250, 45\cdot\sin 250\rangle$

$\langle -15, -42\rangle$

$\langle 212, 64\rangle$

$\boxed{\begin{array}{l} m = 221 \\ \theta = 17^\circ \end{array}}$

Ex.4 The sled dogs have gone mad! A Sled is being pulled in 4 direction and no one know where the sled will end up. You need to help! Dog 1 is pulling due east at 15N, dog 2 is pulling due south at 12N, dog 3 is pulling due west at 13N, and dog 4 is pulling due north at 18N. Determine the direction and the magnitude of the resultant force.

110.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$r = 1 + 2\sin\theta$										

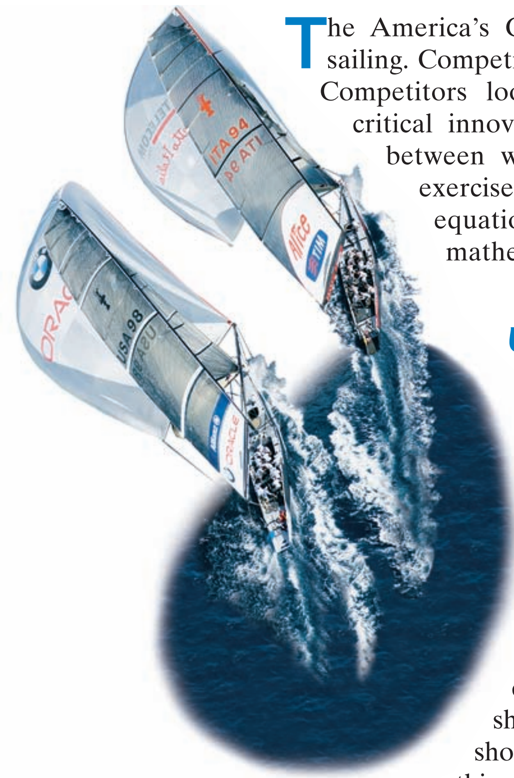
111.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 4\sin 2\theta$									

Section 6.4 Graphs of Polar Equations

Objectives

1. Use point plotting to graph polar equations.
2. Use symmetry to graph polar equations.



The America's Cup is the supreme event in ocean sailing. Competition is fierce and the costs are huge. Competitors look to mathematics to provide the critical innovation that can make the difference between winning and losing. In this section's exercise set, you will see how graphs of polar equations play a role in sailing faster using mathematics.

Using Polar Grids to Graph Polar Equations

Recall that a **polar equation** is an equation whose variables are r and θ. The **graph of a polar equation** is the set of all points whose polar coordinates satisfy the equation. We use **polar grids** like the one shown in **Figure 6.31** to graph polar equations. The grid consists of circles with centers at the pole. This polar grid shows five such circles. A polar grid also shows lines passing through the pole. In this grid, each line represents an angle for which we know the exact values of the trigonometric functions.

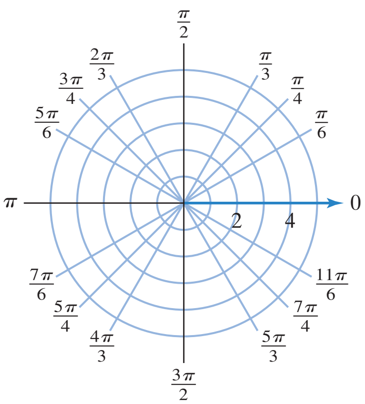


Figure 6.31 A polar coordinate grid

Many polar coordinate grids show more circles and more lines through the pole than in **Figure 6.31**. See if your campus bookstore has paper with polar grids and use the polar graph paper throughout this section.

1 Use point plotting to graph polar equations.

Graphing a Polar Equation by Point Plotting

One method for graphing a polar equation such as $r = 4\cos\theta$ is the **point-plotting method**. First, we make a table of values that satisfy the equation. Next, we plot these ordered pairs as points in the polar coordinate system. Finally, we connect the points with a smooth curve. This often gives us a picture of all ordered pairs (r, θ) that satisfy the equation.

EXAMPLE 1 **Graphing an Equation Using the Point-Plotting Method**

Graph the polar equation $r = 4\cos\theta$ with θ in radians.

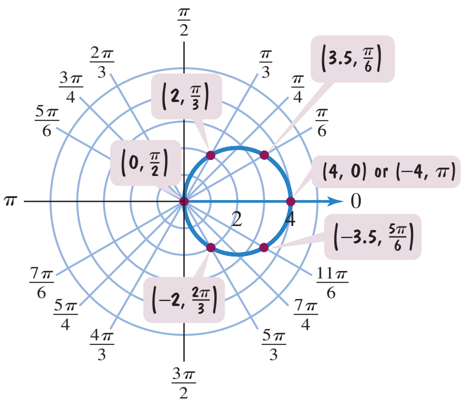


Figure 6.32 The graph of $r = 4 \cos \theta$

Solution We construct a partial table of coordinates for $r = 4 \cos \theta$ using multiples of $\frac{\pi}{6}$. Then we plot the points and join them with a smooth curve, as shown in **Figure 6.32**.

θ	$r = 4 \cos \theta$	(r, θ)
0	$4 \cos 0 = 4 \cdot 1 = 4$	$(4, 0)$
$\frac{\pi}{6}$	$4 \cos \frac{\pi}{6} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \approx 3.5$	$\left(3.5, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$4 \cos \frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2$	$\left(2, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$4 \cos \frac{\pi}{2} = 4 \cdot 0 = 0$	$\left(0, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$4 \cos \frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$	$\left(-2, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$4 \cos \frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} \approx -3.5$	$\left(-3.5, \frac{5\pi}{6}\right)$
π	$4 \cos \pi = 4(-1) = -4$	$(-4, \pi)$
Values of r repeat.		

The graph of $r = 4 \cos \theta$ in **Figure 6.32** looks like a circle of radius 2 whose center is at the point $(x, y) = (2, 0)$. We can verify this observation by changing the polar equation to a rectangular equation.

$r = 4 \cos \theta$	This is the given polar equation.
$r^2 = 4r \cos \theta$	Multiply both sides by r.
$x^2 + y^2 = 4x$	Convert to rectangular coordinates: $r^2 = x^2 + y^2$ and $r \cos \theta = x$.
$x^2 - 4x + y^2 = 0$	Subtract $4x$ from both sides.
$x^2 - 4x + 4 + y^2 = 4$	Complete the square on x: $\frac{1}{2}(-4) = -2$ and $(-2)^2 = 4$. Add 4 to both sides.
$(x - 2)^2 + y^2 = 2^2$	Factor.

This last equation is the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, with radius r and center at (h, k). Thus, the radius is 2 and the center is at $(h, k) = (2, 0)$.

In general, circles have simpler equations in polar form than in rectangular form.

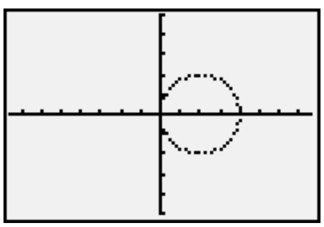
Technology

A graphing utility can be used to obtain the graph of a polar equation. Use the polar mode with angle measure in radians. You must enter the minimum and maximum values for θ and an increment setting for θ, called θ step. θ step determines the number of points that the graphing utility will plot. Make θ step relatively small so that a significant number of points are plotted.

Shown is the graph of $r = 4 \cos \theta$ in a $[-7.5, 7.5, 1]$ by $[-5, 5, 1]$ viewing rectangle with

$$\theta \min = 0$$
$$\theta \max = 2\pi$$
$$\theta \text{ step} = \frac{\pi}{48}.$$

A square setting was used.



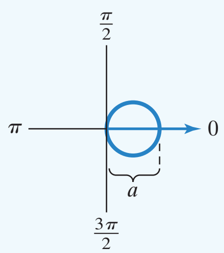
Circles in Polar Coordinates

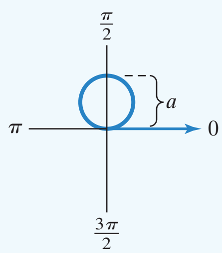
The graphs of

$$r = a \cos \theta \qquad \text{and} \qquad r = a \sin \theta$$

are circles.

$$r = a \cos \theta \qquad \text{and} \qquad r = a \sin \theta$$





Check Point 1 Graph the equation $r = 4 \sin \theta$ with θ in radians. Use multiples of $\frac{\pi}{6}$ from 0 to π to generate coordinates for points (r, θ).

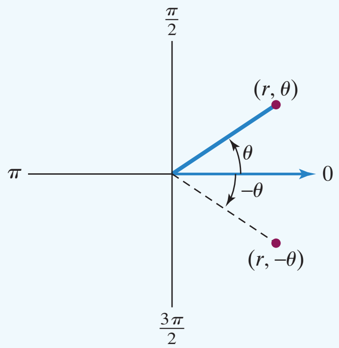
2 Use symmetry to graph polar equations.

Graphing a Polar Equation Using Symmetry

If the graph of a polar equation exhibits symmetry, you may be able to graph it more quickly. Three types of symmetry can be helpful.

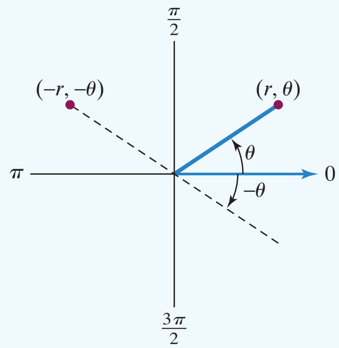
Tests for Symmetry in Polar Coordinates

Symmetry with Respect to the Polar Axis (x-Axis)



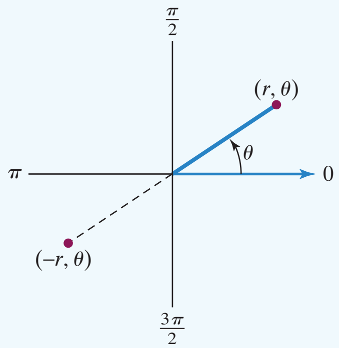
Replace θ with $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis.

Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$ (y-Axis)



Replace (r, θ) with $(-r, -\theta)$. If an equivalent equation results, the graph is symmetric with respect to $\theta = \frac{\pi}{2}$.

Symmetry with Respect to the Pole (Origin)



Replace r with $-r$. If an equivalent equation results, the graph is symmetric with respect to the pole.

If a polar equation passes a symmetry test, then its graph exhibits that symmetry. By contrast, if a polar equation fails a symmetry test, then its graph *may or may not* have that kind of symmetry. Thus, the graph of a polar equation may have a symmetry even if it fails a test for that particular symmetry. Nevertheless, the symmetry tests are useful. If we detect symmetry, we can obtain a graph of the equation by plotting fewer points.

EXAMPLE 2 Graphing a Polar Equation Using Symmetry

Check for symmetry and then graph the polar equation:

$$r = 1 - \cos \theta.$$

Solution We apply each of the tests for symmetry.

Polar Axis: Replace θ with $-\theta$ in $r = 1 - \cos \theta$:

$r = 1 - \cos(-\theta)$ Replace θ with $-\theta$ in $r = 1 - \cos \theta$.

$r = 1 - \cos \theta$ The cosine function is even: $\cos(-\theta) = \cos \theta$.

Because the polar equation does not change when θ is replaced with $-\theta$, the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace (r, θ) with $(-r, -\theta)$ in $r = 1 - \cos \theta$:

$-r = 1 - \cos(-\theta)$ Replace r with $-r$ and θ with $-\theta$ in $r = 1 - \cos \theta$.

$-r = 1 - \cos \theta$ $\cos(-\theta) = \cos \theta$.

$r = \cos \theta - 1$ Multiply both sides by -1.

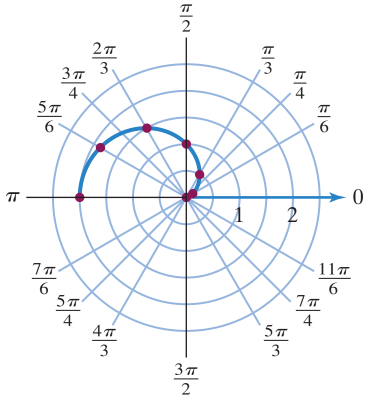


Figure 6.33(a) Graphing $r = 1 - \cos\theta$ for $0 \le \theta \le \pi$

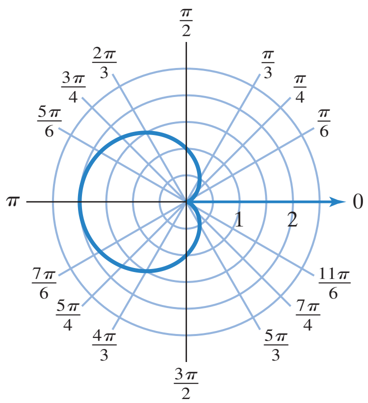


Figure 6.33(b) A complete graph of $r = 1 - \cos\theta$

Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when (r, θ) is replaced with $(-r, -\theta)$, the equation fails this symmetry test. The graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r with $-r$ in $r = 1 - \cos\theta$:

$$-r = 1 - \cos\theta \quad \text{Replace } r \text{ with } -r \text{ in } r = 1 - \cos\theta.$$
$$r = \cos\theta - 1 \quad \text{Multiply both sides by } -1.$$

Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when r is replaced with $-r$, the equation fails this symmetry test. The graph may or may not be symmetric with respect to the pole.

Now we are ready to graph $r = 1 - \cos\theta$. Because the period of the cosine function is 2π, we need not consider values of θ beyond 2π. Recall that we discovered the graph of the equation $r = 1 - \cos\theta$ has symmetry with respect to the polar axis. Because the graph has this symmetry, we can obtain a complete graph by plotting fewer points. Let's start by finding the values of r for values of θ from 0 to π.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	0.13	0.5	1	1.5	1.87	2

The values for r and θ are shown in the table. These values can be obtained using your calculator or possibly with the TABLE feature on some graphing calculators. The points in the table are plotted in **Figure 6.33(a)**. Examine the graph. Keep in mind that the graph must be symmetric with respect to the polar axis. Thus, if we reflect the graph in **Figure 6.33(a)** about the polar axis, we will obtain a complete graph of $r = 1 - \cos\theta$. This graph is shown in **Figure 6.33(b)**.

Check Point 2 Check for symmetry and then graph the polar equation:

$$r = 1 + \cos\theta.$$

EXAMPLE 3 Graphing a Polar Equation

Graph the polar equation: $r = 1 + 2\sin\theta$.

Solution We first check for symmetry.

$$r = 1 + 2\sin\theta$$

Polar Axis	The Line $\theta = \frac{\pi}{2}$	The Pole
Replace θ with $-\theta$.	Replace (r, θ) with $(-r, -\theta)$.	Replace r with $-r$.
$r = 1 + 2\sin(-\theta)$	$-r = 1 + 2\sin(-\theta)$	$-r = 1 + 2\sin\theta$
$r = 1 + 2(-\sin\theta)$	$-r = 1 - 2\sin\theta$	$r = -1 - 2\sin\theta$
$r = 1 - 2\sin\theta$	$r = -1 + 2\sin\theta$	

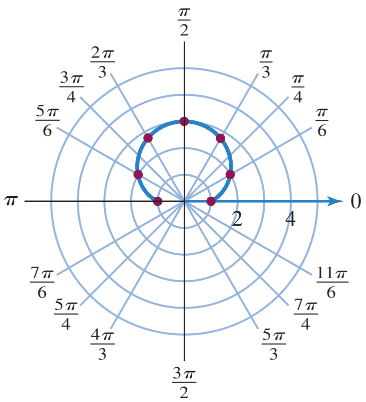
None of these equations are equivalent to $r = 1 + 2\sin\theta$. Thus, the graph may or may not have each of these kinds of symmetry.

Now we are ready to graph $r = 1 + 2\sin\theta$. Because the period of the sine function is 2π, we need not consider values of θ beyond 2π. We identify points on the graph of $r = 1 + 2\sin\theta$ by assigning values to θ and calculating the corresponding values of r. The values for r and θ are in the tables above **Figure 6.34(a)**, **Figure 6.34(b)**, **and Figure 6.34(c)**. The complete graph of $r = 1 + 2\sin\theta$ is shown in **Figure 6.34(c)**. The inner loop indicates that the graph passes through the pole twice.

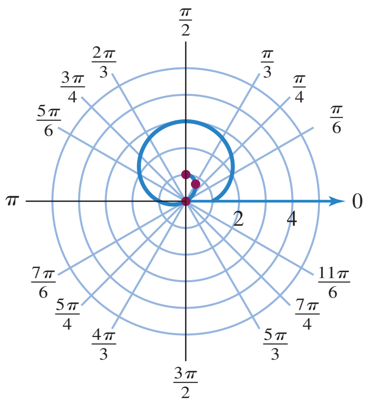
$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\boldsymbol{r}$	1	2	2.73	3	2.73	2	1

$\boldsymbol{\theta}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\boldsymbol{r}$	0	-0.73	-1

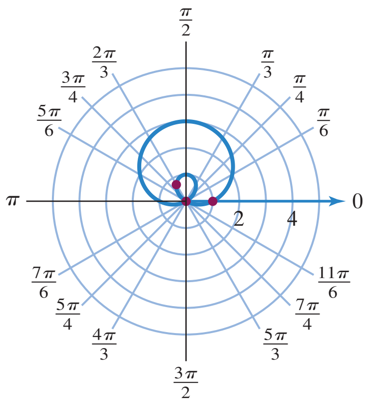
$\boldsymbol{\theta}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{r}$	-0.73	0	1



(a) The graph of $r = 1 + 2\sin\theta$ for $0 \le \theta \le \pi$



(b) The graph of $r = 1 + 2\sin\theta$ for $0 \le \theta \le \frac{3\pi}{2}$



(c) The complete graph of $r = 1 + 2\sin\theta$ for $0 \le \theta \le 2\pi$

Figure 6.34 Graphing $r = 1 + 2\sin\theta$

Although the polar equation $r = 1 + 2\sin\theta$ failed the test for symmetry with respect to the line $\theta = \frac{\pi}{2}$ (the y-axis), its graph in **Figure 6.34(c)** reveals this kind of symmetry.

We're not quite sure if the polar graph in **Figure 6.34(c)** looks like a snail. However, the graph is called a *limaçon*, pronounced "LEE-ma-sohn," which is a French word for snail. Limaçons come with and without inner loops.

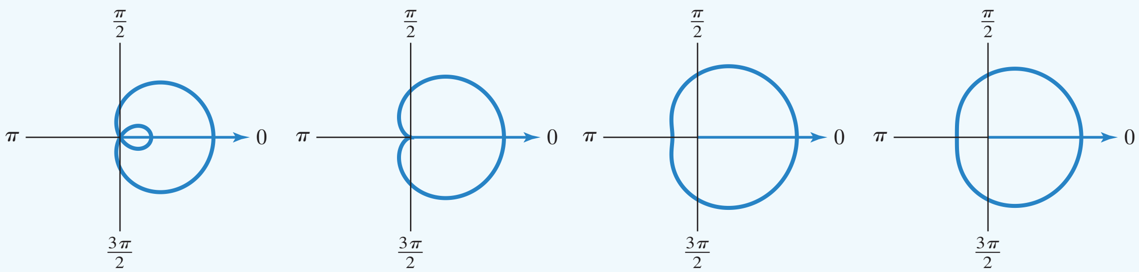
Limaçons

The graphs of

$$r = a + b\sin\theta, \quad r = a - b\sin\theta,$$
$$r = a + b\cos\theta, \quad r = a - b\cos\theta, \quad a > 0, b > 0$$

are called **limaçons**. The ratio $\frac{a}{b}$ determines a limaçon's shape.

Inner loop if $\frac{a}{b} < 1$ | Heart-shaped if $\frac{a}{b} = 1$ and called cardioids | Dimpled with no inner loop if $1 < \frac{a}{b} < 2$ | No dimple and no inner loop if $\frac{a}{b} \ge 2$



Check Point 3 Graph the polar equation: $r = 1 - 2\sin\theta$.

EXAMPLE 4 Graphing a Polar Equation

Graph the polar equation: $r = 4 \sin 2\theta$.

Solution We first check for symmetry.

$$r = 4 \sin 2\theta$$

Polar Axis	**The Line $\theta = \frac{\pi}{2}$**	**The Pole**
Replace θ with $-\theta$. $r = 4 \sin 2(-\theta)$ $r = 4 \sin(-2\theta)$ $r = -4 \sin 2\theta$ Equation changes and fails this symmetry test.	Replace (r, θ) with $(-r, -\theta)$. $-r = 4 \sin 2(-\theta)$ $-r = 4 \sin(-2\theta)$ $-r = -4 \sin 2\theta$ $r = 4 \sin 2\theta$ Equation does not change.	Replace r with $-r$. $-r = 4 \sin 2\theta$ $r = -4 \sin 2\theta$ Equation changes and fails this symmetry test.

Thus, we can be sure that the graph is symmetric with respect to $\theta = \frac{\pi}{2}$. The graph may or may not be symmetric with respect to the polar axis or the pole.

Now we are ready to graph $r = 4 \sin 2\theta$. In **Figure 6.35**, we plot points on the graph of $r = 4 \sin 2\theta$ using values of θ from 0 to π and the corresponding values of r. These coordinates are shown in the tables below.

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\boldsymbol{r}$	0	3.46	4	3.46	0

$\boldsymbol{\theta}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\boldsymbol{r}$	-3.46	-4	-3.46	0

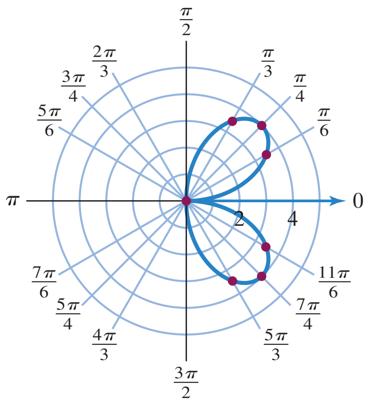


Figure 6.35 The graph of $r = 4 \sin 2\theta$ for $0 \le \theta \le \pi$

Now we can use symmetry with respect to the line $\theta = \frac{\pi}{2}$ (the y-axis) to complete the graph. By reflecting the graph in **Figure 6.35** about the y-axis, we obtain the complete graph of $r = 4 \sin 2\theta$ from 0 to 2π. The graph is shown in **Figure 6.36**.

Although the polar equation $r = 4 \sin 2\theta$ failed the tests for symmetry with respect to the polar axis (the x-axis) and the pole (the origin), its graph in **Figure 6.36** reveals all three types of symmetry.

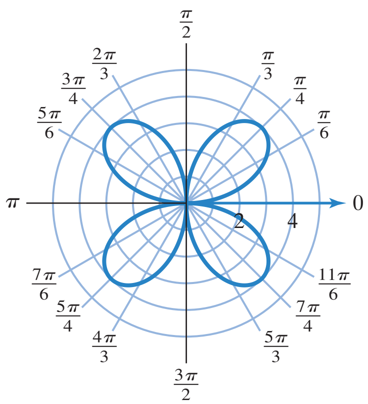


Figure 6.36 The graph of $r = 4 \sin 2\theta$ for $0 \le \theta \le 2\pi$

The curve in **Figure 6.36** is called a **rose with four petals**. We can use a trigonometric equation to confirm the four angles that give the location of the petal points. The petal points of $r = 4 \sin 2\theta$ are located at values of θ for which $r = 4$ or $r = -4$.

$4 \sin 2\theta = 4$ or $4 \sin 2\theta = -4$ — Use $r = 4 \sin 2\theta$ and set r equal to 4 or -4.

$\sin 2\theta = 1$ $\quad$ $\sin 2\theta = -1$ — Divide both sides by 4.

$2\theta = \frac{\pi}{2} + 2n\pi$ $\quad$ $2\theta = \frac{3\pi}{2} + 2n\pi$ — Solve for 2θ, where n is any integer.

$\theta = \frac{\pi}{4} + n\pi$ $\quad$ $\theta = \frac{3\pi}{4} + n\pi$ — Divide both sides by 2 and solve for θ.

If $n = 0$, $\theta = \frac{\pi}{4}$. If $n = 1$, $\theta = \frac{5\pi}{4}$.

If $n = 0$, $\theta = \frac{3\pi}{4}$. If $n = 1$, $\theta = \frac{7\pi}{4}$.

Figure 6.36 confirms that the four angles giving the locations of the petal points are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

Technology

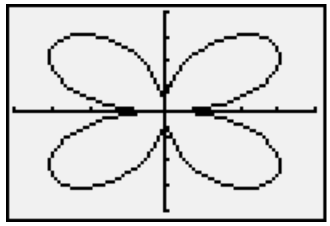
The graph of

$$r = 4 \sin 2\theta$$

was obtained using a $[-4, 4, 1]$ by $[-4, 4, 1]$ viewing rectangle and

$$\theta \text{ min} = 0, \quad \theta \text{ max} = 2\pi,$$

$$\theta \text{ step} = \frac{\pi}{48}.$$



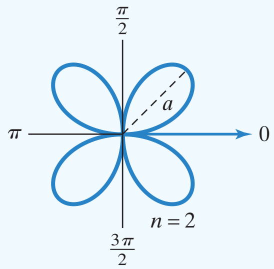
Rose Curves

The graphs of

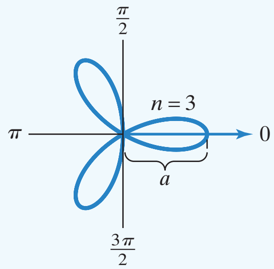
$$r = a \sin n\theta \quad \text{and} \quad r = a \cos n\theta, \quad a \neq 0,$$

are called **rose curves**. If n is even, the rose has $2n$ petals. If n is odd, the rose has n petals.

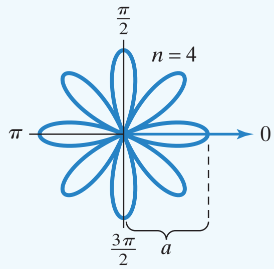
$r = a \sin 2\theta$
Rose curve with 4 petals



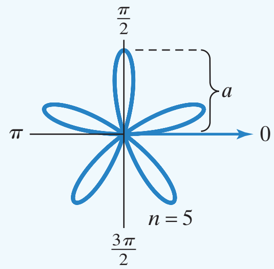
$r = a \cos 3\theta$
Rose curve with 3 petals



$r = a \cos 4\theta$
Rose curve with 8 petals



$r = a \sin 5\theta$
Rose curve with 5 petals



Check Point 4 Graph the polar equation: $r = 3 \cos 2\theta$.

EXAMPLE 5 Graphing a Polar Equation

Graph the polar equation: $r^2 = 4 \sin 2\theta$.

Solution We first check for symmetry.

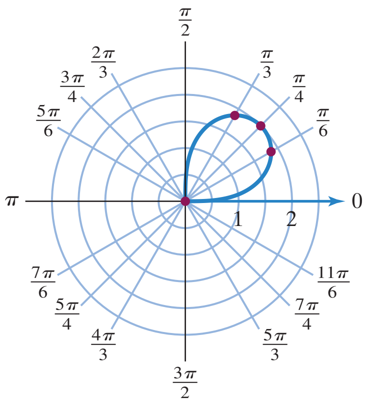
$$r^2 = 4 \sin 2\theta$$

Polar Axis	**The Line $\theta = \frac{\pi}{2}$**	**The Pole**
Replace θ with $-\theta$. $r^2 = 4 \sin 2(-\theta)$ $r^2 = 4 \sin(-2\theta)$ $r^2 = -4 \sin 2\theta$ Equation changes and fails this symmetry test.	Replace (r, θ) with $(-r, -\theta)$. $(-r)^2 = 4 \sin 2(-\theta)$ $r^2 = 4 \sin(-2\theta)$ $r^2 = -4 \sin 2\theta$ Equation changes and fails this symmetry test.	Replace r with $-r$. $(-r)^2 = 4 \sin 2\theta$ $r^2 = 4 \sin 2\theta$ Equation does not change.

Thus, we can be sure that the graph is symmetric with respect to the pole. The graph may or may not be symmetric with respect to the polar axis or the line $\theta = \frac{\pi}{2}$.

Now we are ready to graph $r^2 = 4 \sin 2\theta$. In **Figure 6.37(a)**, we plot points on the graph by using values of θ from 0 to $\frac{\pi}{2}$ and the corresponding values of r. These coordinates are shown in the table to the left of **Figure 6.37(a)**. Notice that the points in **Figure 6.37(a)** are shown for $r \geq 0$. Because the graph is symmetric with respect to the pole, we can reflect the graph in **Figure 6.37(a)** about the pole and obtain the graph in **Figure 6.37(b)**.

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\boldsymbol{r}$	0	± 1.9	± 2	± 1.9	0



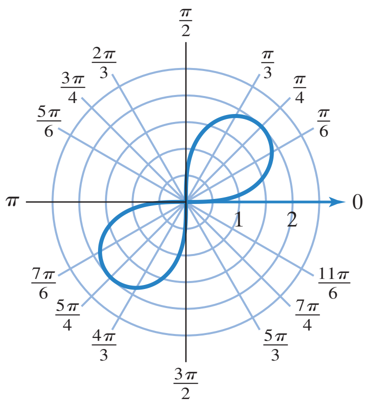


Figure 6.37 Graphing $r^2 = 4 \sin 2\theta$

(a) The graph of $r^2 = 4 \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$ and $r \geq 0$

(b) Using symmetry with respect to the pole on the graph of $r^2 = 4 \sin 2\theta$

Does **Figure 6.37(b)** show a complete graph of $r^2 = 4 \sin 2\theta$ or do we need to continue graphing for angles greater than $\frac{\pi}{2}$? If θ is in quadrant II, 2θ is in quadrant III or IV, where $\sin 2\theta$ is negative. Thus, $4 \sin 2\theta$ is negative. However, $r^2 = 4 \sin 2\theta$ and r^2 cannot be negative. The same observation applies to quadrant IV. This means that there are no points on the graph in quadrants II or IV. Thus, **Figure 6.37(b)** shows the complete graph of $r^2 = 4 \sin 2\theta$.

The curve in **Figure 6.37(b)** is shaped like a propeller and is called a *lemniscate.*

Lemniscates

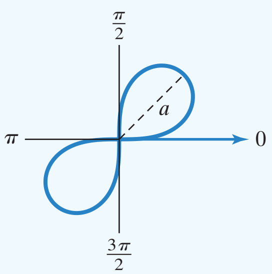
The graphs of

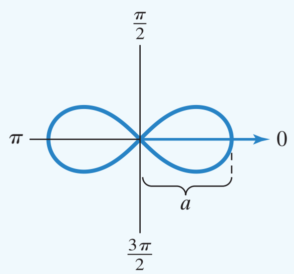
$$r^2 = a^2 \sin 2\theta \quad \text{and} \quad r^2 = a^2 \cos 2\theta, \quad a \neq 0$$

are called **lemniscates**.

$r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole.

$r^2 = a^2 \cos 2\theta$ is symmetric with respect to the polar axis, $\theta = \frac{\pi}{2}$, and the pole.





Check Point 5 Graph the polar equation: $r^2 = 4 \cos 2\theta$.

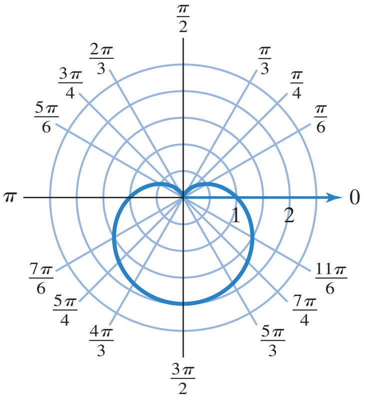
Exercise Set 6.4

Practice Exercises

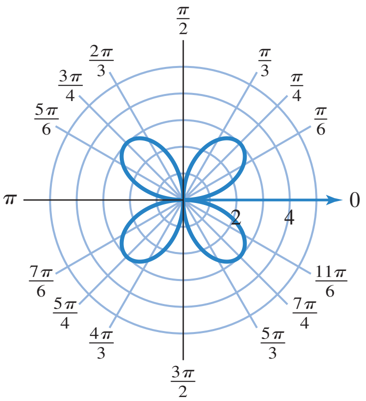
In Exercises 1–6, the graph of a polar equation is given. Select the polar equation for each graph from the following options.

$$r = 2\sin\theta,\quad r = 2\cos\theta,\quad r = 1 + \sin\theta,$$
$$r = 1 - \sin\theta,\quad r = 3\sin 2\theta,\quad r = 3\sin 3\theta$$

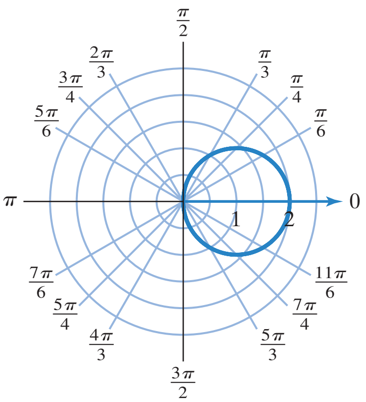
1.



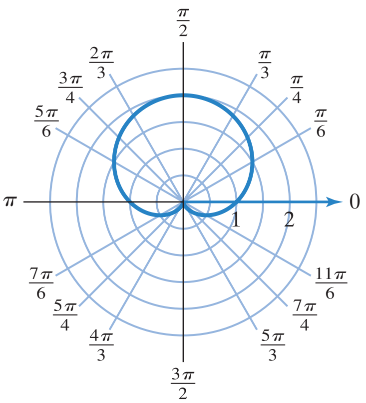
2.



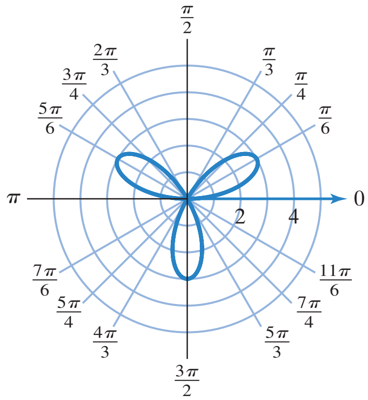
3.



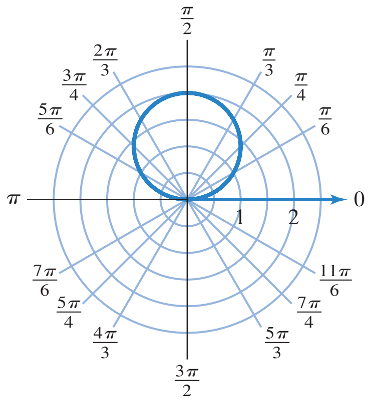
4.



5.



6.



In Exercises 7–12, test for symmetry with respect to

a. *the polar axis.* **b.** *the line* $\theta = \frac{\pi}{2}$. **c.** *the pole.*

7. $r = \sin\theta$ **8.** $r = \cos\theta$

9. $r = 4 + 3\cos\theta$ **10.** $r = 2\cos 2\theta$

11. $r^2 = 16\cos 2\theta$ **12.** $r^2 = 16\sin 2\theta$

In Exercises 13–34, test for symmetry and then graph each polar equation.

13. $r = 2\cos\theta$ **14.** $r = 2\sin\theta$

15. $r = 1 - \sin\theta$ **16.** $r = 1 + \sin\theta$

17. $r = 2 + 2\cos\theta$ **18.** $r = 2 - 2\cos\theta$

19. $r = 2 + \cos\theta$ **20.** $r = 2 - \sin\theta$

21. $r = 1 + 2\cos\theta$ **22.** $r = 1 - 2\cos\theta$

23. $r = 2 - 3\sin\theta$ **24.** $r = 2 + 4\sin\theta$

25. $r = 2\cos 2\theta$ **26.** $r = 2\sin 2\theta$

27. $r = 4\sin 3\theta$ **28.** $r = 4\cos 3\theta$

29. $r^2 = 9\cos 2\theta$ **30.** $r^2 = 9\sin 2\theta$

31. $r = 1 - 3\sin\theta$ **32.** $r = 3 + \sin\theta$

33. $r\cos\theta = -3$ **34.** $r\sin\theta = 2$

Practice Plus

In Exercises 35–44, test for symmetry and then graph each polar equation.

35. $r = \cos\frac{\theta}{2}$ **36.** $r = \sin\frac{\theta}{2}$

37. $r = \sin\theta + \cos\theta$ **38.** $r = 4\cos\theta + 4\sin\theta$

39. $r = \dfrac{1}{1 - \cos\theta}$ **40.** $r = \dfrac{2}{1 - \cos\theta}$

41. $r = \sin\theta\cos^2\theta$

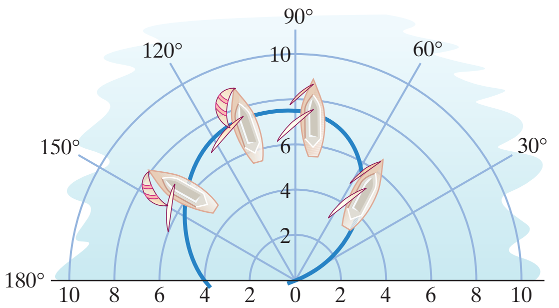
42. $r = \dfrac{3\sin 2\theta}{\sin^3\theta + \cos^3\theta}$

43. $r = 2 + 3\sin 2\theta$

44. $r = 2 - 4\cos 2\theta$

Application Exercises

In Exercise Set 6.3, we considered an application in which sailboat racers look for a sailing angle to a 10-knot wind that produces maximum sailing speed. This situation is now represented by the polar graph in the figure shown. Each point (r, θ) on the graph gives the sailing speed, r, in knots, at an angle θ to the 10-knot wind. Use this information to solve Exercises 45–49.



45. What is the speed, to the nearest knot, of a sailboat sailing at a 60° angle to the wind?

46. What is the speed, to the nearest knot, of a sailboat sailing at a 120° angle to the wind?

47. What is the speed, to the nearest knot, of a sailboat sailing at a 90° angle to the wind?

48. What is the speed, to the nearest knot, of a sailboat sailing at a 180° angle to the wind?

49. What angle to the wind produces the maximum sailing speed? What is the speed? Round the angle to the nearest five degrees and the speed to the nearest half knot.

Writing in Mathematics

50. What is a polar equation?

51. What is the graph of a polar equation?

52. Describe how to graph a polar equation.

53. Describe the test for symmetry with respect to the polar axis.

54. Describe the test for symmetry with respect to the line $\theta = \dfrac{\pi}{2}$.

55. Describe the test for symmetry with respect to the pole.

56. If an equation fails the test for symmetry with respect to the polar axis, what can you conclude?

Technology Exercises

Use the polar mode of a graphing utility with angle measure in radians to solve Exercises 57–88. Unless otherwise indicated, use θ min $= 0$, θ max $= 2\pi$, and θ step $= \dfrac{\pi}{48}$. If you are not pleased with the quality of the graph, experiment with smaller values for θ step. However, if θ step is extremely small, it can take your graphing utility a long period of time to complete the graph.

57. Use a graphing utility to verify any six of your hand-drawn graphs in Exercises 13–34.

In Exercises 58–75, use a graphing utility to graph the polar equation.

58. $r = 4\cos 5\theta$

59. $r = 4\sin 5\theta$

60. $r = 4\cos 6\theta$

61. $r = 4\sin 6\theta$

62. $r = 2 + 2\cos\theta$

63. $r = 2 + 2\sin\theta$

64. $r = 4 + 2\cos\theta$

65. $r = 4 + 2\sin\theta$

66. $r = 2 + 4\cos\theta$

67. $r = 2 + 4\sin\theta$

68. $r = \dfrac{3}{\sin\theta}$

69. $r = \dfrac{3}{\cos\theta}$

70. $r = \cos\dfrac{3}{2}\theta$

71. $r = \cos\dfrac{5}{2}\theta$

72. $r = 3\sin\left(\theta + \dfrac{\pi}{4}\right)$

73. $r = 2\cos\left(\theta - \dfrac{\pi}{4}\right)$

74. $r = \dfrac{1}{1 - \sin\theta}$

75. $r = \dfrac{1}{3 - 2\sin\theta}$

In Exercises 76–78, find the smallest interval for θ starting with θ min $= 0$ so that your graphing utility graphs the given polar equation exactly once without retracing any portion of it.

76. $r = 4\sin\theta$

77. $r = 4\sin 2\theta$

78. $r^2 = 4\sin 2\theta$

In Exercises 79–82, use a graphing utility to graph each butterfly curve. Experiment with the range setting, particularly θ step, to produce a butterfly of the best possible quality.

79. $r = \cos^2 5\theta + \sin 3\theta + 0.3$

80. $r = \sin^4 4\theta + \cos 3\theta$

81. $r = \sin^5\theta + 8\sin\theta\cos^3\theta$

82. $r = 1.5^{\sin\theta} - 2.5\cos 4\theta + \sin^7\dfrac{\theta}{15}$ (Use θ min $= 0$ and θ max $= 20\pi$.)

83. Use a graphing utility to graph $r = \sin n\theta$ for $n = 1, 2, 3, 4, 5$, and 6. Use a separate viewing screen for each of the six graphs. What is the pattern for the number of loops that occur corresponding to each value of n? What is happening to the shape of the graphs as n increases? For each graph, what is the smallest interval for θ so that the graph is traced only once?

84. Repeat Exercise 83 for $r = \cos n\theta$. Are your conclusions the same as they were in Exercise 83?

85. Use a graphing utility to graph $r = 1 + 2\sin n\theta$ for $n = 1, 2, 3, 4, 5$, and 6. Use a separate viewing screen for each of the six graphs. What is the pattern for the number of large and small petals that occur corresponding to each value of n? How are the large and small petals related when n is odd and when n is even?

86. Repeat Exercise 85 for $r = 1 + 2\cos n\theta$. Are your conclusions the same as they were in Exercise 85?

87. Graph the spiral $r = \theta$. Use a $[-30, 30, 1]$ by $[-30, 30, 1]$ viewing rectangle. Let θ min $= 0$ and θ max $= 2\pi$, then θ min $= 0$ and θ max $= 4\pi$, and finally θ min $= 0$ and θ max $= 8\pi$.

88. Graph the spiral $r = \dfrac{1}{\theta}$. Use a $[-1, 1, 1]$ by $[-1, 1, 1]$ viewing rectangle. Let θ min $= 0$ and θ max $= 2\pi$, then θ min $= 0$ and θ max $= 4\pi$, and finally θ min $= 0$ and θ max $= 8\pi$.

Critical Thinking Exercises

Make Sense? *In Exercises 89–92, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

89. I'm working with a polar equation that failed the symmetry test with respect to $\theta = \frac{\pi}{2}$, so my graph will not have this kind of symmetry.

90. The graph of my limaçon exhibits none of the three kinds of symmetry discussed in this section.

91. There are no points on my graph of $r^2 = 9\cos 2\theta$ for which $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

92. I'm graphing a polar equation in which for every value of θ there is exactly one corresponding value of r, yet my polar coordinate graph fails the vertical line for functions.

In Exercises 93–94, graph r_1 and r_2 in the same polar coordinate system. What is the relationship between the two graphs?

93. $r_1 = 4\cos 2\theta,\ r_2 = 4\cos 2\left(\theta - \frac{\pi}{4}\right)$

94. $r_1 = 2\sin 3\theta,\ r_2 = 2\sin 3\left(\theta + \frac{\pi}{6}\right)$

95. Describe a test for symmetry with respect to the line $\theta = \frac{\pi}{2}$ in which r is not replaced.

Preview Exercises

Exercises 96–98 will help you prepare for the material covered in the next section. Refer to Section 2.1 if you need to review the basics of complex numbers. In each exercise, perform the indicated operation and write the result in the standard form $a + bi$.

96. $(1 + i)(2 + 2i)$

97. $\left(-1 + i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right)$

98. $\frac{2 + 2i}{1 + i}$

Chapter 6 Mid-Chapter Check Point

What You Know: We learned to solve oblique triangles using the Laws of Sines $\left(\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$ and Cosines $(a^2 = b^2 + c^2 - 2bc\cos A)$. We applied the Law of Sines to SAA, ASA, and SSA (the ambiguous case) triangles. We applied the Law of Cosines to SAS and SSS triangles. We found areas of SAS triangles $\left(\text{area} = \frac{1}{2}bc\sin A\right)$ and SSS triangles (Heron's formula: $\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$, s is $\frac{1}{2}$ the perimeter). We used the polar coordinate system to plot points and represented them in multiple ways. We used the relations between polar and rectangular coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2,\ \tan\theta = \frac{y}{x}$$

to convert points and equations from one coordinate system to the other. Finally, we used point plotting and symmetry to graph polar equations.

In Exercises 1–6, solve each triangle. Round lengths to the nearest tenth and angle measures to the nearest degree. If no triangle exists, state "no triangle." If two triangles exist, solve each triangle.

1. $A = 32°, B = 41°, a = 20$

2. $A = 42°, a = 63, b = 57$

3. $A = 65°, a = 6, b = 7$

4. $B = 110°, a = 10, c = 16$

5. $C = 42°, a = 16, c = 13$

6. $a = 5.0, b = 7.2, c = 10.1$

In Exercises 7–8, find the area of the triangle having the given measurements. Round to the nearest square unit.

7. $C = 36°, a = 5$ feet, $b = 7$ feet

8. $a = 7$ meters, $b = 9$ meters, $c = 12$ meters

9. Two trains leave a station on different tracks that make an angle of 110° with the station as vertex. The first train travels at an average rate of 50 miles per hour and the second train travels at an average rate of 40 miles per hour. How far apart, to the nearest tenth of a mile, are the trains after 2 hours?

10. Two fire-lookout stations are 16 miles apart, with station B directly east of station A. Both stations spot a fire on a mountain to the south. The bearing from station A to the fire is S56°E. The bearing from station B to the fire is S23°W. How far, to the nearest tenth of a mile, is the fire from station A?

11. A tree that is perpendicular to the ground sits on a straight line between two people located 420 feet apart. The angles of elevation from each person to the top of the tree measure 50° and 66°, respectively. How tall, to the nearest tenth of a foot, is the tree?

In Exercises 12–15, convert the given coordinates to the indicated ordered pair.

12. $\left(-3, \frac{5\pi}{4}\right)$ to (x, y)

13. $\left(6, -\frac{\pi}{2}\right)$ to (x, y)

14. $\left(2, -2\sqrt{3}\right)$ to (r, θ)

15. $(-6, 0)$ to (r, θ)

In Exercises 16–17, plot each point in polar coordinates. Then find another representation (r, θ) of this point in which:

a. $r > 0,\quad 2\pi < \theta < 4\pi.$

b. $r < 0,\quad 0 < \theta < 2\pi.$

c. $r > 0, -2\pi < \theta < 0.$

16. $\left(4, \frac{3\pi}{4}\right)$

17. $\left(\frac{5}{2}, \frac{\pi}{2}\right)$