Section 1.3: Transformations of Graphs

- Vertical and Horizontal Shifts of Graphs
- Reflecting, Stretching, and Shrinking of Graphs
- Combining Transformations

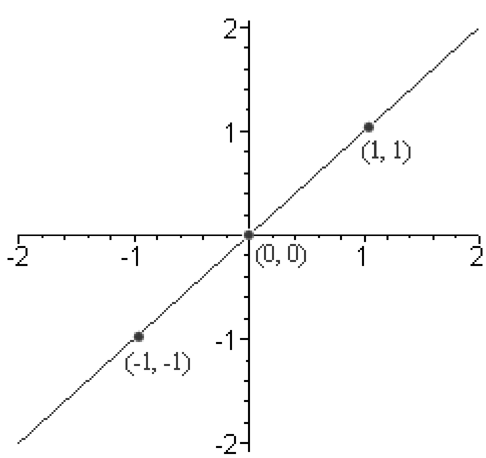
Vertical and Horizontal Shifts of Graphs

Graphs of Some Basic Functions:

$f(x) = x$

domain: $(-\infty, \infty)$

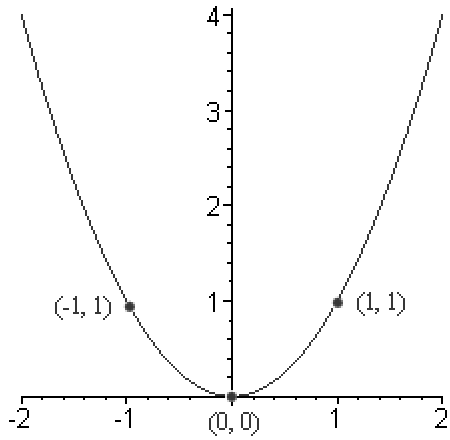
range: $(-\infty, \infty)$



$f(x) = x^2$

domain: $(-\infty, \infty)$

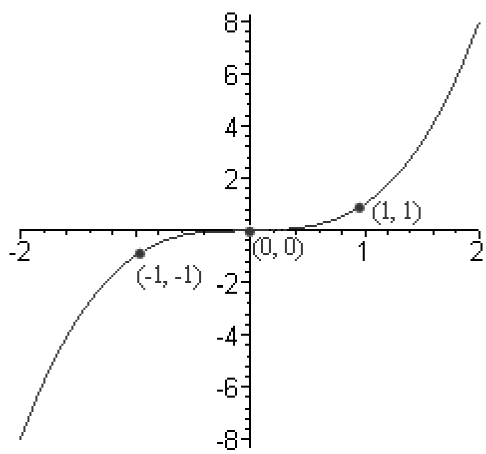
range: $[0, \infty)$



$f(x) = x^3$

domain: $(-\infty, \infty)$

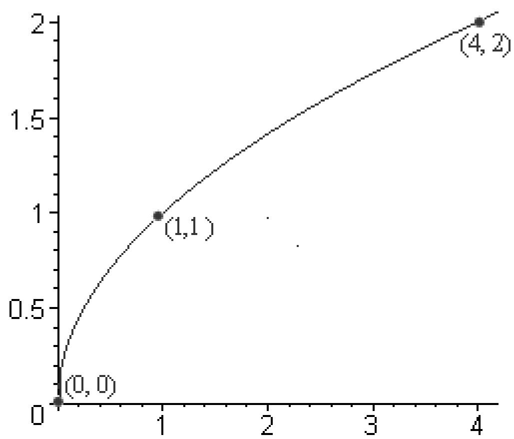
range: $(-\infty, \infty)$



$f(x) = \sqrt{x}$

domain : $[0, \infty)$

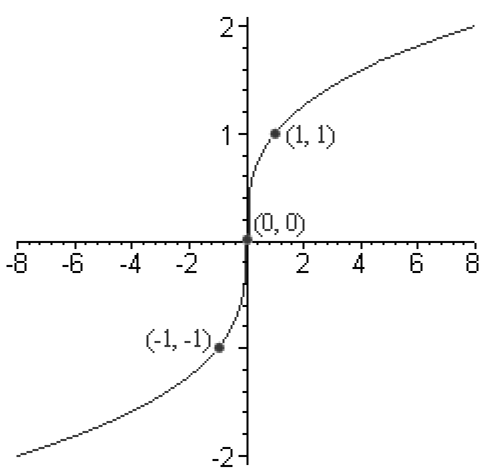
range : $[0, \infty)$



$f(x) = \sqrt[3]{x}$

domain : $(-\infty, \infty)$

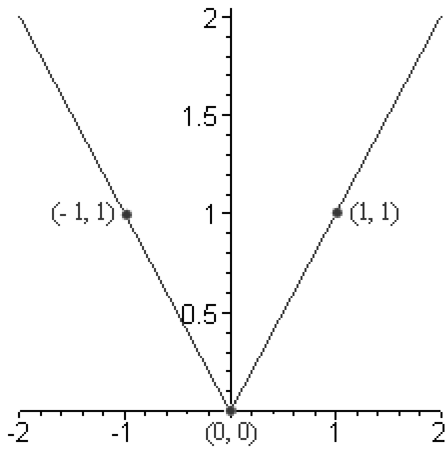
range : $(-\infty, \infty)$



$f(x) = |x|$

domain : $(-\infty, \infty)$

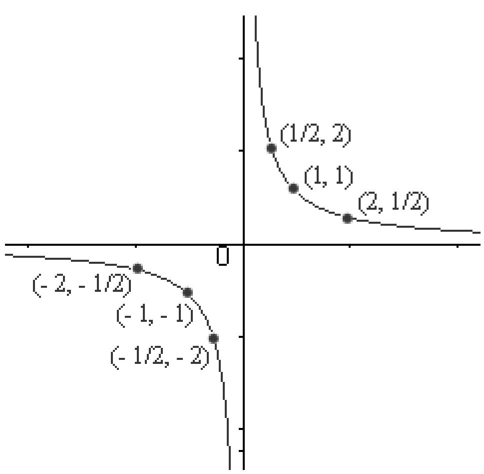
range : $[0, \infty)$



$f(x) = \dfrac{1}{x}$

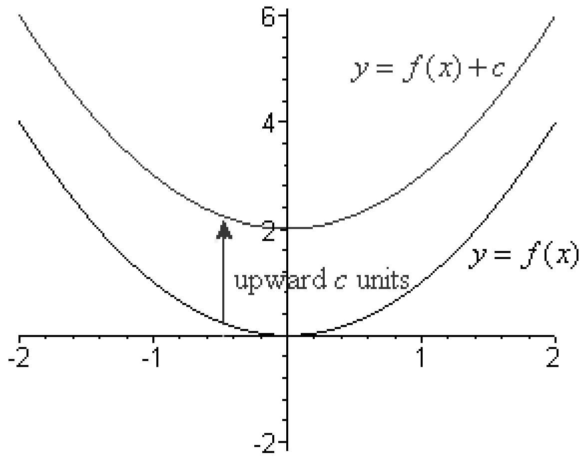
domain : $(-\infty, 0) \cup (0, \infty)$

range : $(-\infty, 0) \cup (0, \infty)$

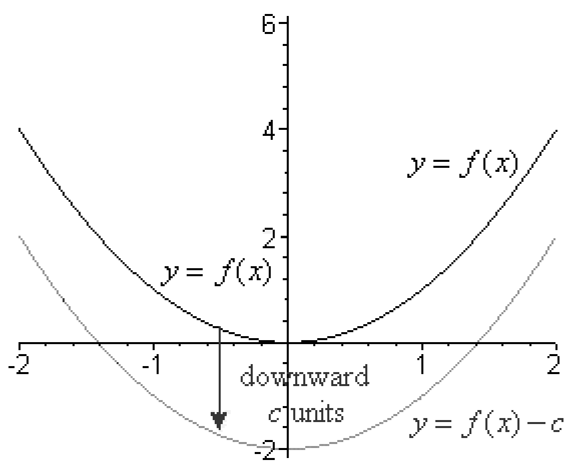


Vertical Shifts of Graphs:

To graph the function $y = f(x) + c$, where $c > 0$, shift the graph of $y = f(x)$ upward c units.

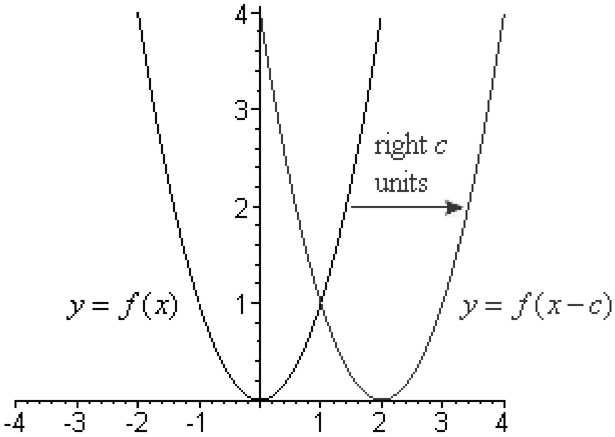


To graph the function $y = f(x) - c$, where $c > 0$, shift the graph of $y = f(x)$ downward c units.

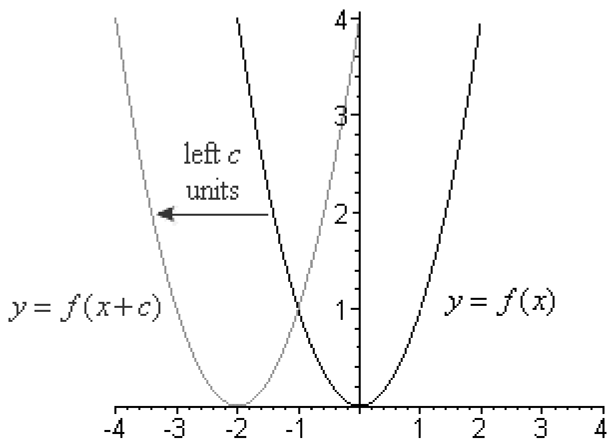


Horizontal Shifts of Graphs:

To graph the function $y = f(x - c)$, where $c > 0$, shift the graph of $y = f(x)$ to the right c units.



To graph the function $y = f(x+c)$, where $c > 0$, shift the graph of $y = f(x)$ to the left c units.

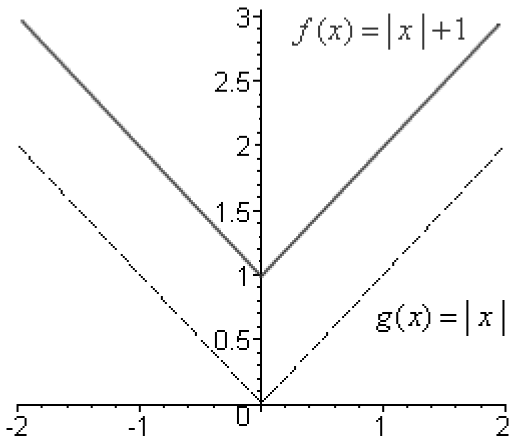


Example:

Sketch the graph of the function $f(x) = |x| + 1$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

Start with the graph of the function $g(x) = |x|$. To graph the function $f(x) = |x| + 1$, shift the graph of g upward 1 unit. (The graph of g is displayed below in a dotted format to serve as a guide in graphing the given function f.)

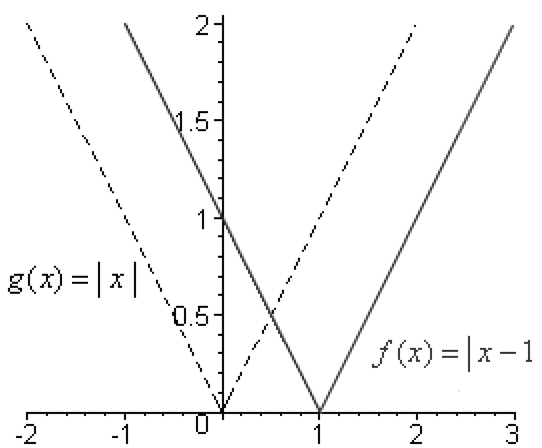


Example:

Sketch the graph of the function $f(x) = |x - 1|$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

Start with the graph of the function $g(x) = |x|$. To graph the function $f(x) = |x - 1|$, shift the graph of g to the right 1 unit. (The graph of g is displayed below in a dotted format to serve as a guide in graphing the given function f.)

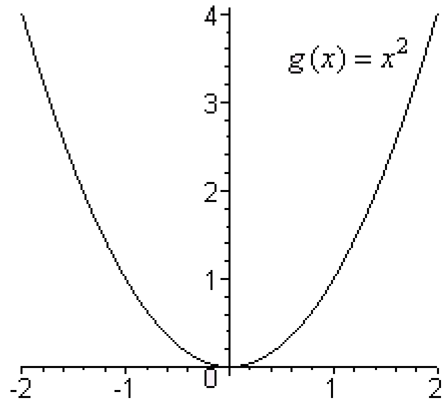


Additional Example 1:

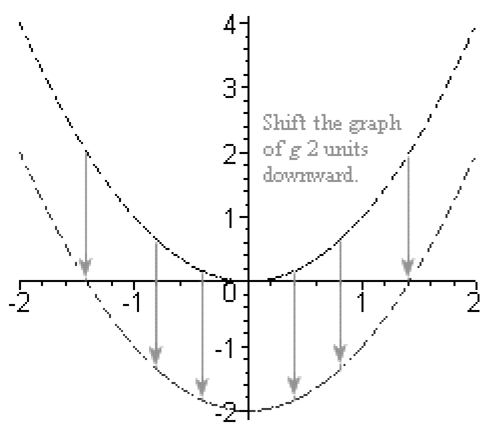
Sketch the graph of the function $f(x) = x^2 - 2$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

Begin with the graph of $g(x) = x^2$ shown below.



To graph the function $f(x) = x^2 - 2$, shift the graph of g 2 units downward.

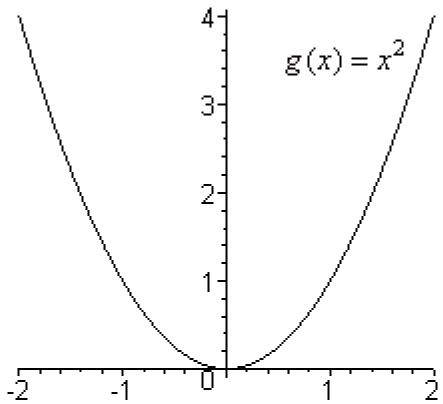


Additional Example 2:

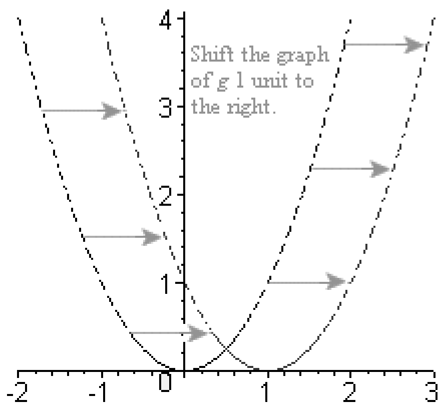
Sketch the graph of the function $f(x) = (x-1)^2$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

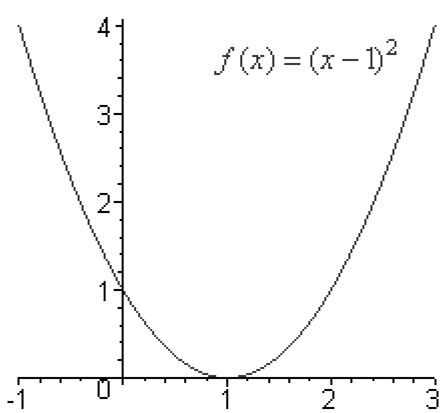
Begin with the graph of $g(x) = x^2$ shown below.



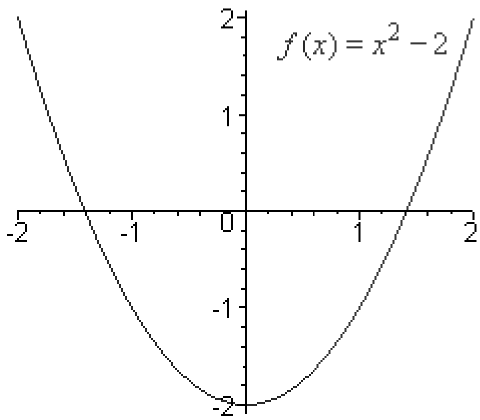
To graph the function $f(x) = (x-1)^2$, shift the graph of g 1 unit to the right.



The graph of the given function is shown below.



The graph of the given function is shown below.

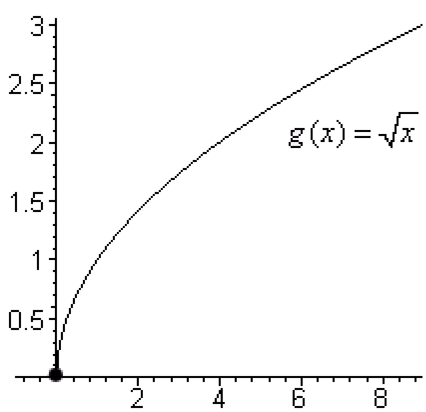


Additional Example 3:

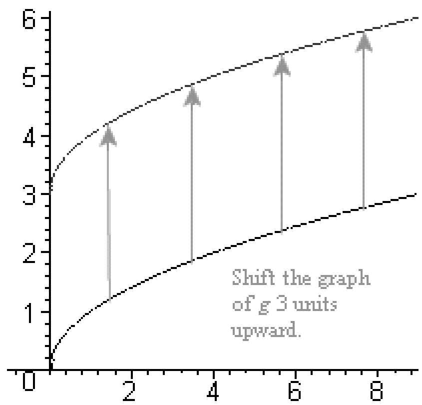
Sketch the graph of the function $f(x) = \sqrt{x} + 3$ Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

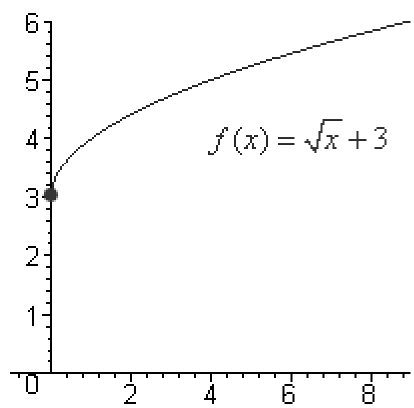
Begin with the graph of $g(x) = \sqrt{x}$ shown below.



To graph the function $f(x) = \sqrt{x} + 3$, shift the graph of g 3 units upward.



The graph of the given function is shown below.

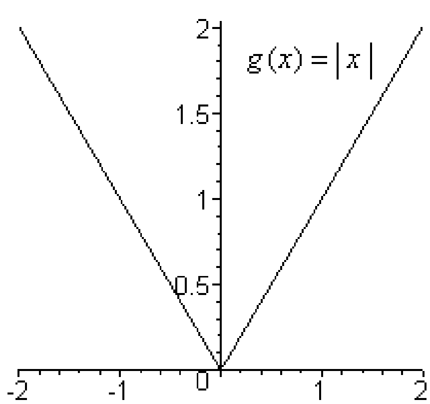


Additional Example 4:

Sketch the graph of the function $f(x) = |x + 3| - 2$ Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

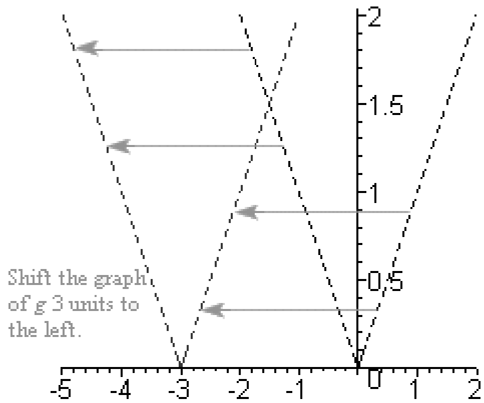
Begin with the graph of $g(x) = |x|$ shown below.



First shift the graph of g 3 units to the left.

$\left[f(x) = |x+3| - 2\right]$

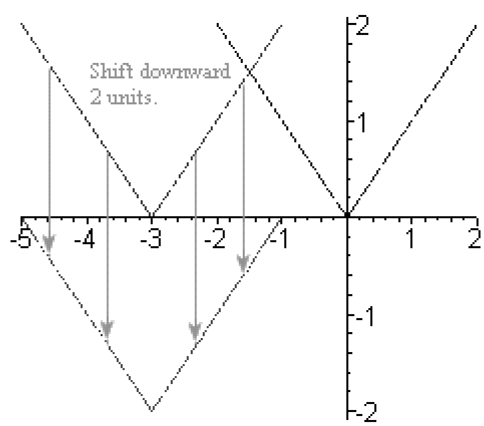
Shift left 3 units.



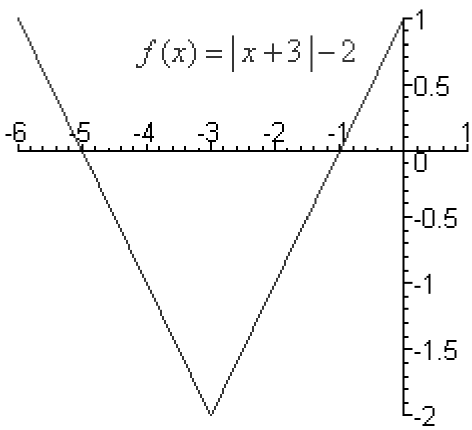
Now shift 2 units downward.

$\left[f(x) = |x+3| - 2\right]$

Shift downward 2 units.

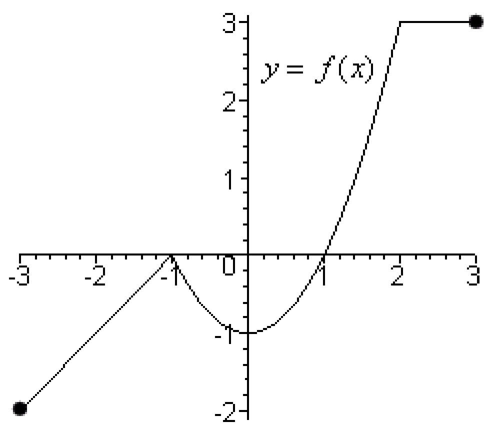


The graph of the given function is shown below.



Additional Example 5:

The graph of a function $y = f(x)$ is shown below. Sketch the graph of the function $y = f(x-1)+2$.

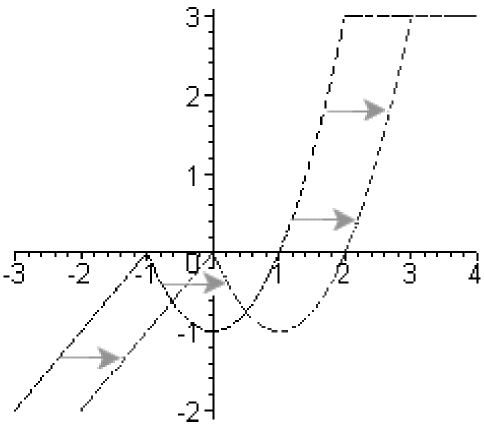


Solution:

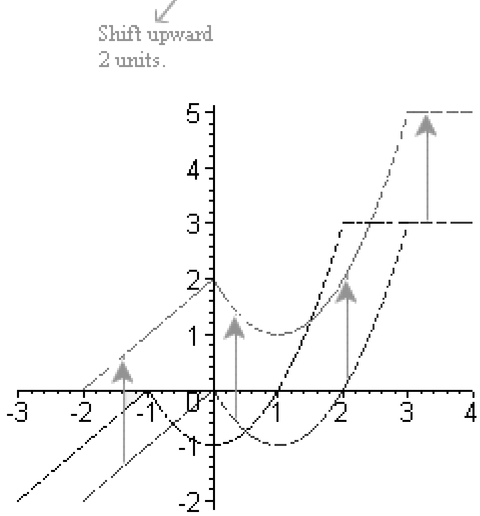
First shift the graph of f 1 unit to the right.

$[y = f(x-1)+2]$

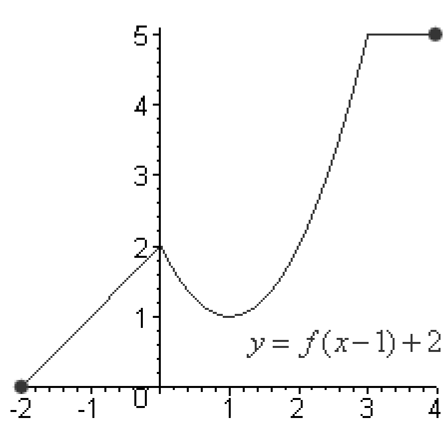
Shift to the right 1 unit.



Now shift upward
2 units.
$[y = f(x-1) + 2]$



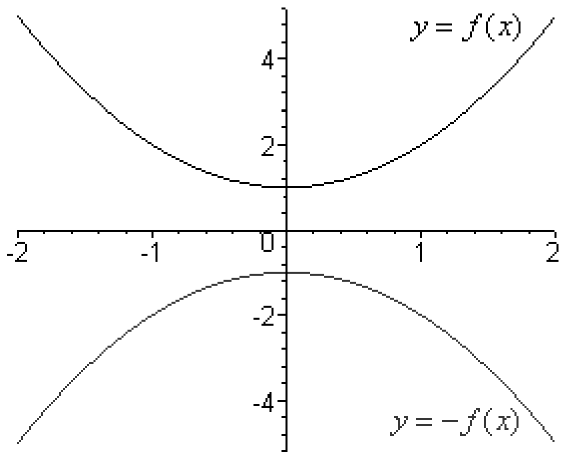
The graph of the given function is shown below.



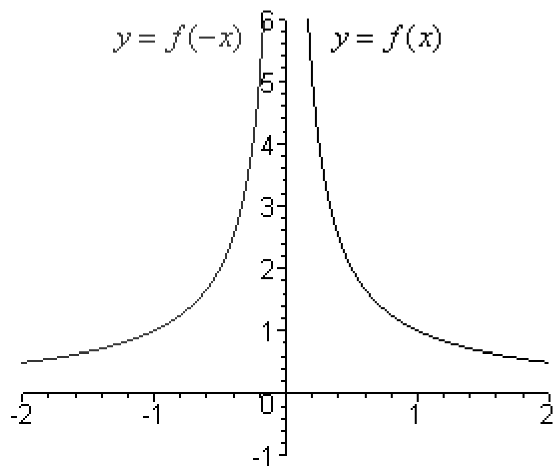
Reflecting, Stretching, and Shrinking of Graphs

Reflecting Graphs:

To graph the function $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

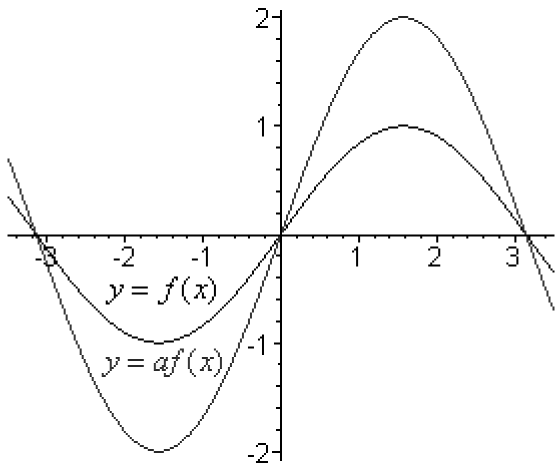


To graph the function $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.

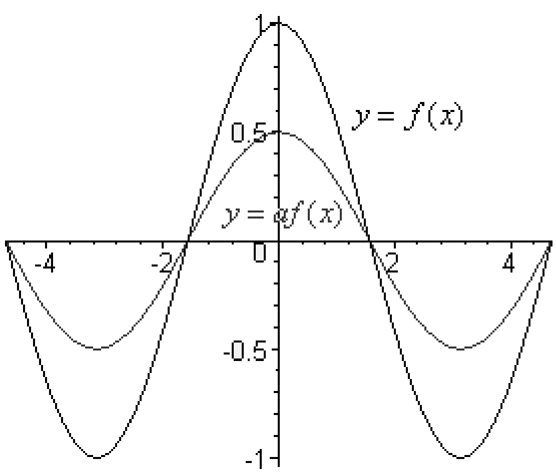


Stretching and Shrinking of Graphs:

To graph the function $y = af(x)$, where $a > 1$, stretch the graph of $y = f(x)$ vertically by a factor of a. (In the graph shown below, $a = 2$.)

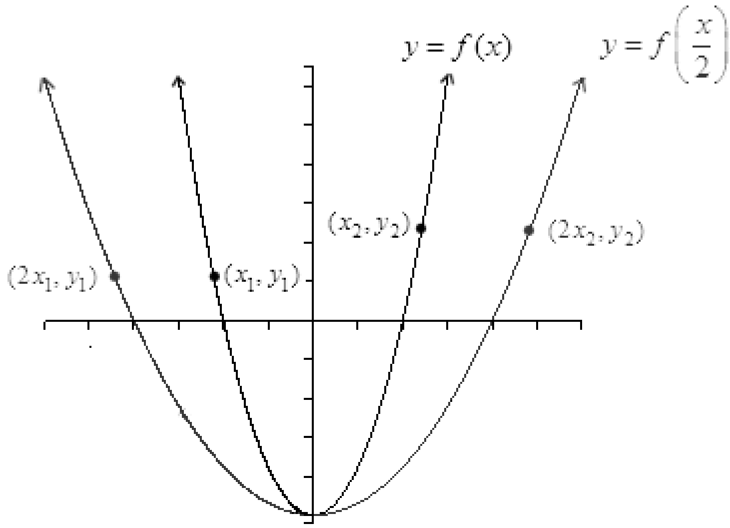


To graph the function $y = af(x)$, where $0 < a < 1$, shrink the graph of $y = f(x)$ vertically by a factor of a. (In the graph shown below, $a = \frac{1}{2}$.)



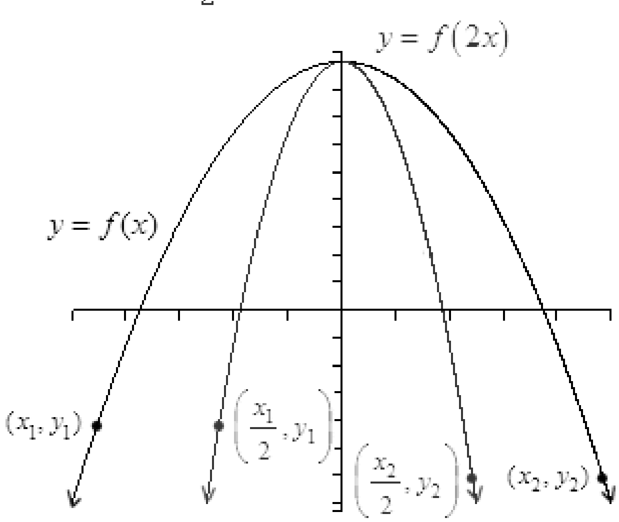
To graph the function $y = f(ax)$, where $0 < a < 1$, <u>stretch</u> the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.

So, for the function $y = f\left(\frac{x}{2}\right)$, <u>stretch</u> the graph of $y = f(x)$ horizontally by a factor of 2. (Note: $a = \frac{1}{2}$ in this example, so $\frac{1}{a} = \frac{1}{\frac{1}{2}} = 2$.)



To graph the function $y = f(ax)$, where $a > 1$, <u>shrink</u> the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{a}$.

So, for the function $y = f(2x)$, <u>shrink</u> the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{2}$.

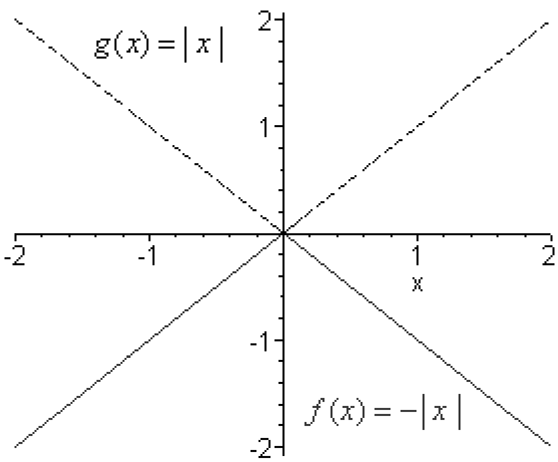


Example:

Sketch the graph of the function $f(x) = -|x|$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

Start with the graph of the function $g(x) = |x|$. To graph the function $f(x) = -|x|$, reflect the graph of g in the x-axis. (The graph of g is displayed below in a dotted format to serve as a guide in graphing the given function f.)

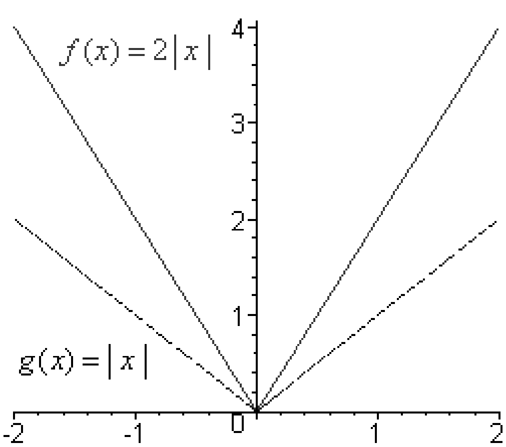


Example:

Sketch the graph of the function $f(x) = 2|x|$. Do not plot points, but instead apply transformations to the graph of a standard function.

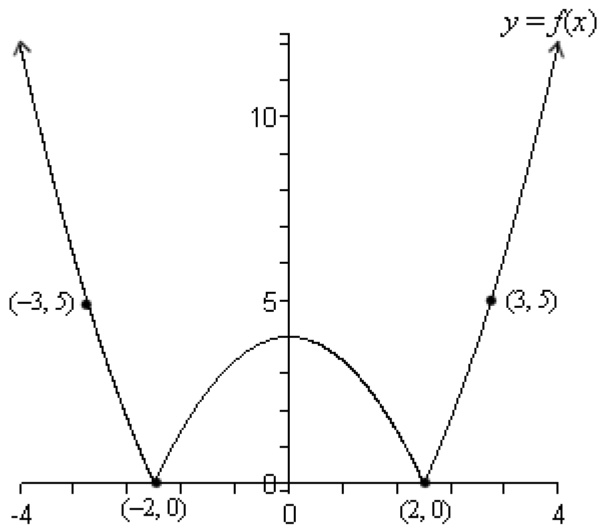
Solution:

Start with the graph of the function $g(x) = |x|$. To graph the function $f(x) = 2|x|$, stretch the graph of g by a factor of 2. (The graph of g is displayed below in a dotted format to serve as a guide in graphing the given function f.)



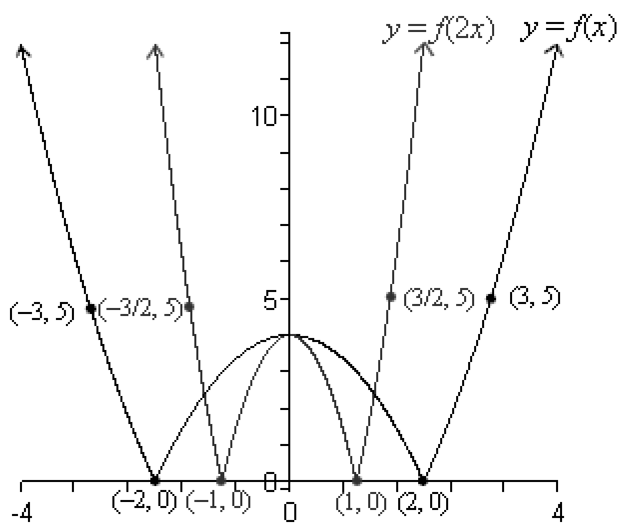
Example:

The graph of a function $y = f(x)$ is shown in the figure below. Use the graph of f to sketch the graph of $y = f(2x)$.



Solution:

To graph the function $y = f(2x)$, shrink the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{2}$.



Example:

Sketch the graph of the function $f(x) = \sqrt{1-x}$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

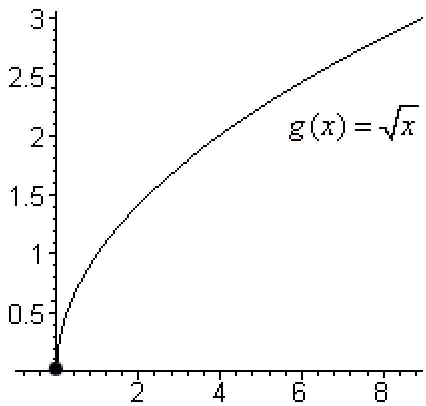
We will show several methods of solution.

Solution 1:

Rewrite the function.

$$f(x) = \sqrt{1-x} = \sqrt{-(x-1)}$$

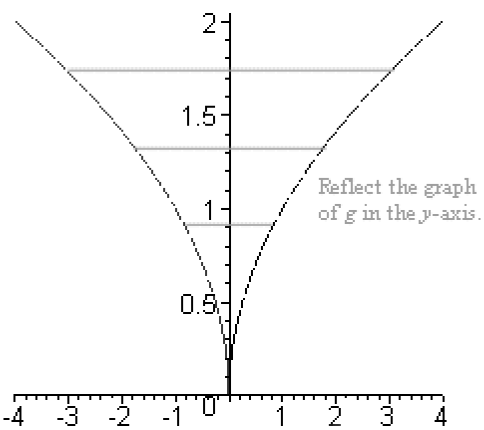
Begin with the graph of $g(x) = \sqrt{x}$ shown below.



First reflect the graph of g in the y-axis.

$f(x) = \sqrt{-(x-1)}$

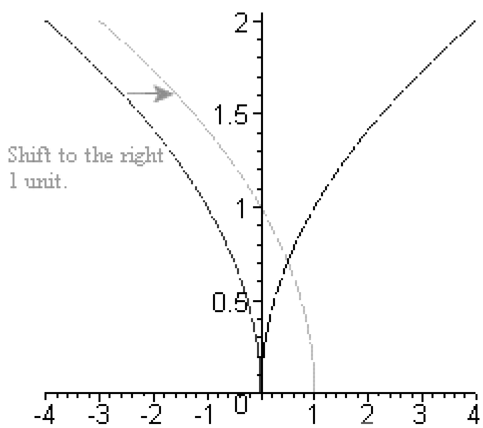
Reflect in the y-axis.



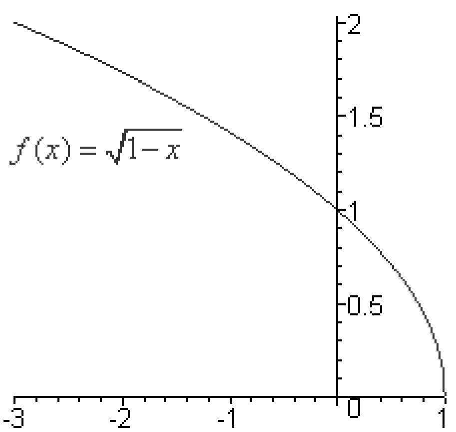
Now shift to the right 1 unit.

$f(x) = \sqrt{-(x-1)}$

Shift to the right 1 unit.



The graph of the given function is shown below.



The first solution uses the following sequence of transformations.

Step 1) $g(x) = \sqrt{x}$

Step 2) $g(-x) = \sqrt{-x} = h(x)$ Reflect in the y-axis.

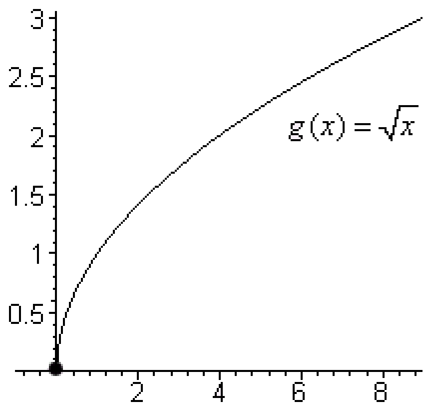
Step 3) $h(x-1) = \sqrt{-(x-1)} = \sqrt{1-x} = f(x)$ Shift to the right 1 unit.

Solution 2:

Rewrite the function.

$$f(x) = \sqrt{1-x} = \sqrt{-x+1}$$

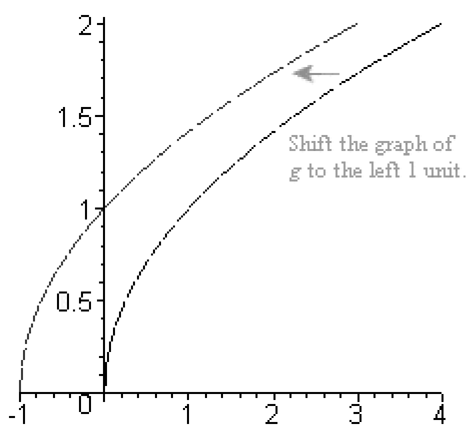
Begin with the graph of $g(x) = \sqrt{x}$ shown below.



First shift the graph
of g to the left 1 unit.

$f(x) = \sqrt{-x\boxed{+1}}$

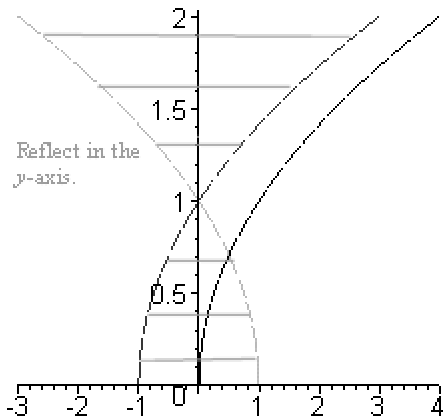
Shift to the left
1 unit.

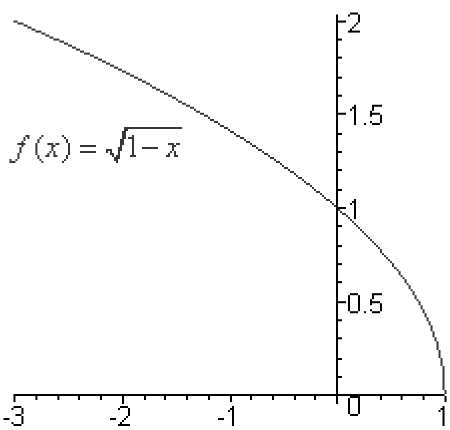


Now reflect in the y-axis.

$f(x) = \sqrt{-x+1}$

Reflect in the y-axis.





The second solution uses the following sequence of transformations.

Step 1) $g(x) = \sqrt{x}$

Step 2) $g(x+1) = \sqrt{x+1} = h(x)$ Shift to the left 1 unit.

Step 3) $h(-x) = \sqrt{-x+1} = \sqrt{1-x} = f(x)$ Reflect in the y-axis..

The example given above shows that different sequences of transformations can lead to the same graph.

Care must be taken in applying transformations. For example, suppose one reflects the graph of $g(x)=\sqrt{x}$ in the y-axis followed by a shift to the left of 1 unit. Then this proposed sequence of transformations will not yield the graph of the function $f(x)=\sqrt{1-x}$ as shown in the steps below.

Step 1) $g(x)=\sqrt{x}$

Step 2) $g(-x)=\sqrt{-x}=h(x)$ Reflect in the y-axis.

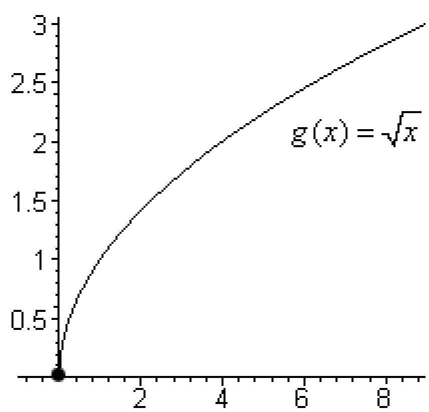
Step 3) $h(x+1)=\sqrt{-(x+1)}=\sqrt{-1-x}\neq f(x)$ Shift to the left 1 unit

Additional Example 1:

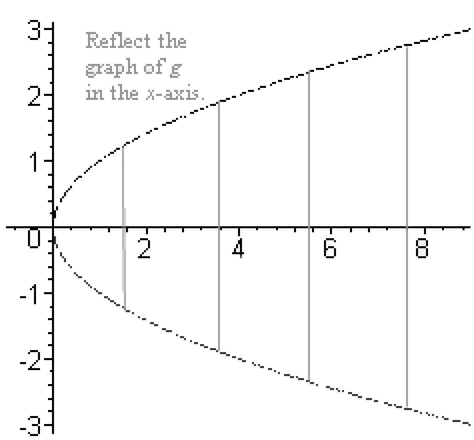
Sketch the graph of the function $f(x)=-\sqrt{x}$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

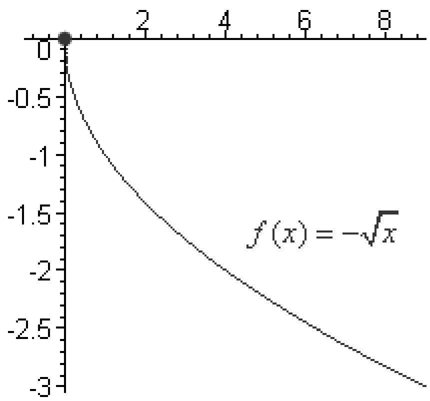
Begin with the graph of $g(x)=\sqrt{x}$ shown below.



To graph the function $f(x)=-\sqrt{x}$, reflect the graph of g in the x-axis.

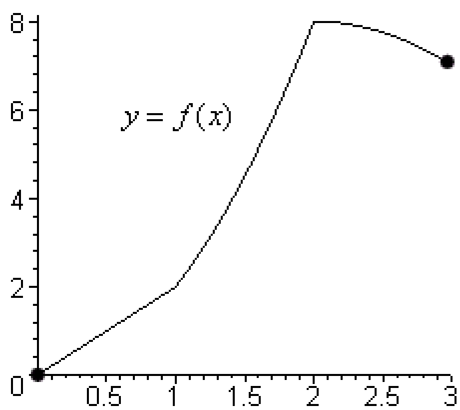


The graph of the given function is shown below.



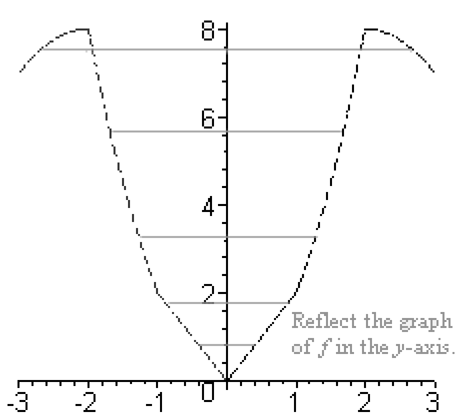
Additional Example 2:

The graph of a function $y = f(x)$ is shown below. Sketch the graph of the function $y = f(-x)$.

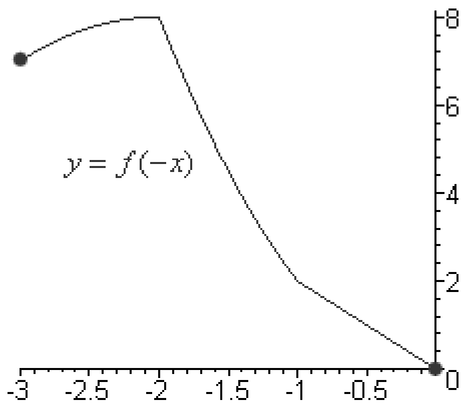


Solution:

To graph the function $y = f(-x)$, reflect the graph of f in the y-axis.



The graph of the given function is shown below.

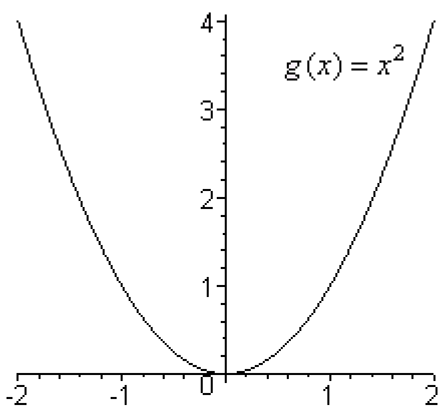


Additional Example 3:

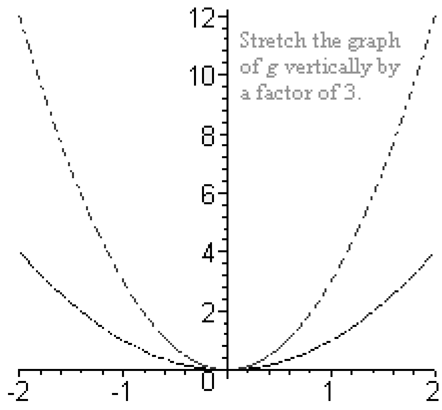
Sketch the graph of the function $f(x) = 3x^2$. Do not plot points, but instead apply transformations to the graph of a standard function.

Solution:

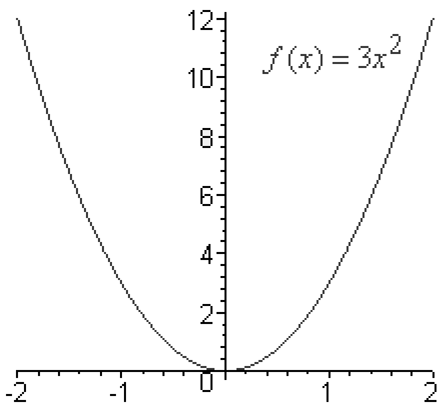
Begin with the graph of $g(x) = x^2$ shown below.



To graph the function $f(x) = 3x^2$, stretch the graph of g vertically by a factor of 3.



The graph of the given function is shown below.



Combining Transformations

Suppose that you want to graph the function $f(x) = 3\sqrt{x+2} - 7$. We can quickly identify from the function that the 'base' function is $g(x) = \sqrt{x}$, and that there has been a vertical stretch with a factor of 3, a shift left of 2 units, and a downward shift of 7 units. If you are graphing this function, does the order matter when you perform the transformations? For example, can you shift down, then do the vertical stretch, then shift left? Or should you first shift left, then shift down, and then perform the vertical stretch? We could come up with many different possibilities for the order of transformations for this problem. In this particular example, the order does matter, and we could get an incorrect graph if we perform certain operations out of order. (There are other cases where the order does not matter, depending on which transformations are used.) It is worth spending some time analyzing the order of transformations – which can be done algebraically, without any trial-and-error in graphing.

First, remember the rules for transformations of functions.
(These are not listed in any recommended order; they are just listed for review.)

RULES FOR TRANSFORMATIONS OF FUNCTIONS

If $f(x)$ is the original function, $a > 0$ and $c > 0$:

Function	**Transformation of the graph of *f* (*x*)**
$f(x)+c$	Shift $f(x)$ upward *c* units
$f(x)-c$	Shift $f(x)$ downward *c* units
$f(x+c)$	Shift $f(x)$ to the left *c* units
$f(x-c)$	Shift $f(x)$ to the right c units
$-f(x)$	Reflect $f(x)$ in the *x*-axis
$f(-x)$	Reflect $f(x)$ in the *y*-axis
$a \cdot f(x),\ a > 1$	Stretch $f(x)$ vertically by a factor of *a*.
$a \cdot f(x),\ 0 < a < 1$	Shrink $f(x)$ vertically by a factor of *a*.
$f(ax),\ a > 1$	Shrink $f(x)$ horizontally by a factor of $\frac{1}{a}$.
$f(ax),\ 0 < a < 1$	Stretch $f(x)$ horizontally by a factor of $\frac{1}{a}$.

Let us look at Examples 1 through 6 below, and we will then look for a pattern as to when the order of transformations matters.

Example Problem 1: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Shift upward 7 units, then right 2 units.

(b) Shift right 2 units, then upward 7 units.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}+7 \quad \rightarrow \quad h(x)=\sqrt{x-2}+7$

Up 7   Right 2

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x-2} \quad \rightarrow \quad h(x)=\sqrt{x-2}+7$

Right 2   Up 7

© Yes, parts (a) and (b) yield the same function.

Example Problem 2: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Stretch vertically by a factor of 2, then shift downward 5 units.

(b) Shift downward 5 units, then stretch vertically by a factor of 2.

© Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=2\sqrt{x} \quad \rightarrow \quad h(x)=2\sqrt{x}-5$

Stretch vertically by a factor of 2   Down 5

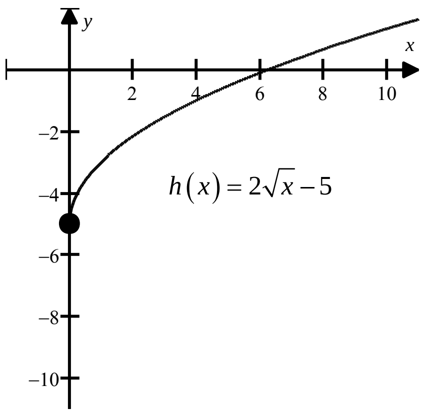
(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}\ -5 \quad \rightarrow \quad h(x)=2\left(\sqrt{x}-5\right)$

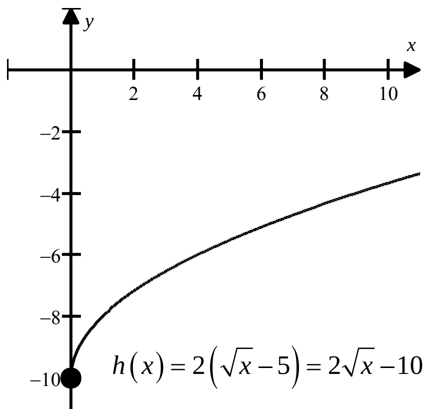
Down 5 — Stretch vertically by a factor of 2

Note: In part (b), $h(x)$ can also be written as $h(x)=2\sqrt{x}-10$.

© No, parts (a) and (b) do not yield the same function, since $2\sqrt{x}-5 \neq 2\sqrt{x}-10$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).





Example Problem 3: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *y*-axis, then shift upward 6 units.

(b) Shift upward 6 units, then reflect in the *y*-axis.

© Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{-x} \quad \rightarrow \quad h(x)=\sqrt{-x}\ +6$

Reflect in the *y*-axis — Up 6

(b) $f(x)=\sqrt{x}$ $\rightarrow$ $g(x)=\sqrt{x}+6$ $\rightarrow$ $h(x)=\sqrt{-x}+6$

Up 6 — Reflect in the *y*-axis

(c) Yes, parts (a) and (b) yield the same function.

Example Problem 4: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *y*-axis, then shift left 2 units.

(b) Shift left 2 units, then reflect in the *y*-axis.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x}$ $\rightarrow$ $g(x)=\sqrt{-x}$ $\rightarrow$ $h(x)=\sqrt{-(x+2)}$

Reflect in the *y*-axis — Left 2

Note: In part (a), $h(x)$ can also be written as $h(x)=\sqrt{-x-2}$.

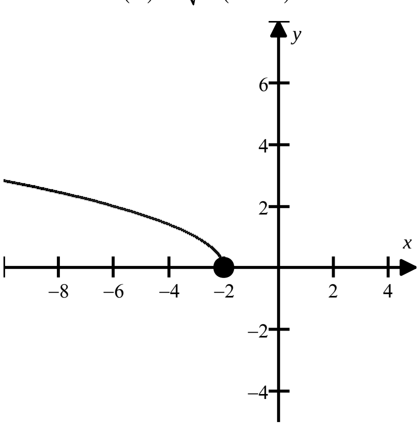
(b) $f(x)=\sqrt{x}$ $\rightarrow$ $g(x)=\sqrt{x+2}$ $\rightarrow$ $h(x)=\sqrt{-x+2}$

Left 2 — Reflect in the *y*-axis

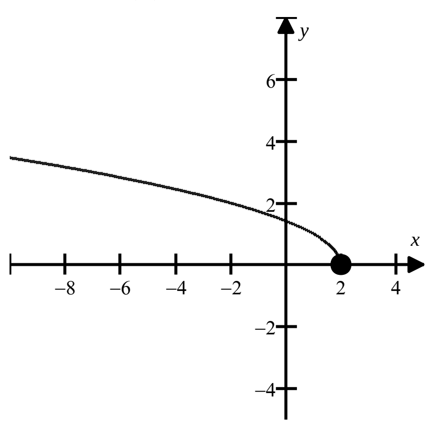
(c) No, parts (a) and (b) do not yield the same function, since $\sqrt{-x-2}\neq\sqrt{-x+2}$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

Part (a): $h(x)=\sqrt{-(x+2)}=\sqrt{-x-2}$



Part (b): $h(x)=\sqrt{-x+2}$



Example Problem 5: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Reflect in the *x*-axis, then shift upward 4 units.

(b) Shift upward 4 units, then reflect in the *x*-axis.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=-\sqrt{x} \quad \rightarrow \quad h(x)=-\sqrt{x}+4$

Reflect in the *x*-axis; Up 4

(b) $f(x)=\sqrt{x} \quad \rightarrow \quad g(x)=\sqrt{x}+4 \quad \rightarrow \quad h(x)=-\left(\sqrt{x}+4\right)$

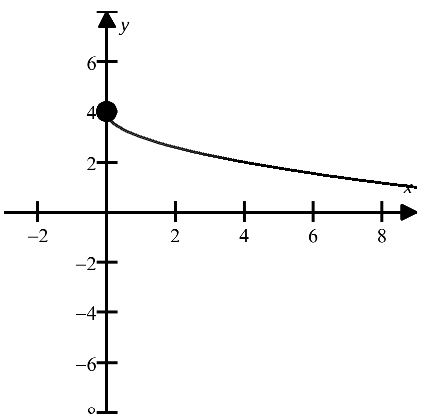
Up 4; Reflect in the *x*-axis

Note: In part (b), $h(x)$ can also be written as $h(x)=-\sqrt{x}-4$.

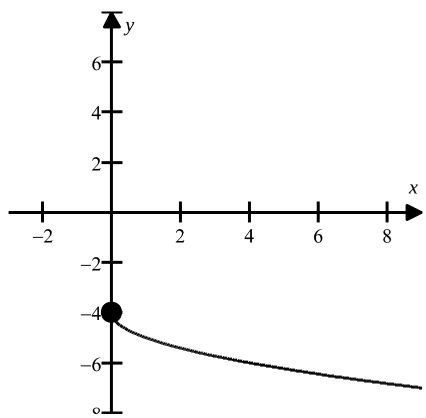
(c) No, parts (a) and (b) do not yield the same function, since $-\sqrt{x}+4 \neq -\sqrt{x}-4$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

Part (a):
$h(x)=-\sqrt{x}+4$



Part (b):
$h(x)=-\left(\sqrt{x}+4\right)=-\sqrt{x}-4$



Example Problem 6: Start with the function $f(x)=\sqrt{x}$, and write the function which results from the given transformations. Then decide if the results from parts (a) and (b) are equivalent.

(a) Shrink horizontally by a factor of $\frac{1}{3}$, then shift right 6 units.

(b) Shift right 6 units, then shrink horizontally by a factor of $\frac{1}{3}$.

(c) Do parts (a) and (b) yield the same function? (You should be able to tell without graphing.)

Solution:

(a) $f(x)=\sqrt{x} \quad \to \quad g(x)=\sqrt{3x} \quad \to \quad h(x)=\sqrt{3(x-6)}$

Shrink horizontally by a factor of $\frac{1}{3}$ (first arrow); Right 6 (second arrow)

Note: In part (a), $h(x)$ can also be written as $h(x)=\sqrt{3x-18}$.

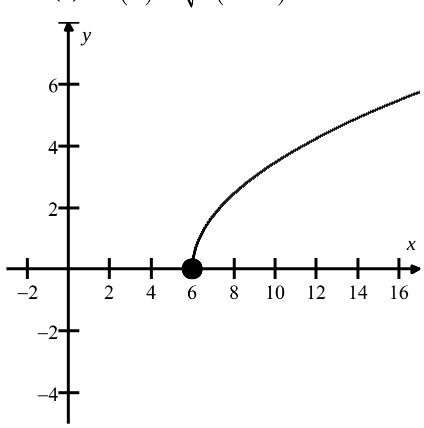
(b) $f(x)=\sqrt{x} \quad \to \quad g(x)=\sqrt{x-6} \quad \to \quad h(x)=\sqrt{3x-6}$

Right 6 (first arrow); Shrink horizontally by a factor of $\frac{1}{3}$ (second arrow)

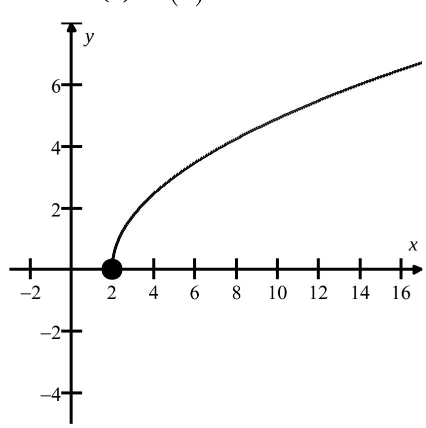
(c) No, parts (a) and (b) do not yield the same function, since $\sqrt{3x-18} \neq \sqrt{3x-6}$.

Both graphs are shown below to emphasize the difference in the final results (but we can see that the above functions are different without graphing the functions).

Part (a): $h(x)=\sqrt{3(x-6)}=\sqrt{3x-18}$



Part (b): $h(x)=\sqrt{3x-6}$



Looking for a Pattern – When Does the Order of Transformations Matter?

When deciding whether the order of the transformations matters, it helps to think about whether a transformation affects the graph vertically (i.e. changes the *y*-values) or horizontally (i.e. changes the *x*-values).

Transformation	Vertical or Horizontal Effect?
Shifting up or down	Vertical
Shifting left or right	Horizontal
Reflecting in the *y*-axis	Horizontal
Reflecting in the *x*-axis	Vertical
Vertical stretching/shrinking	Vertical
Horizontal stretching/shrinking	Horizontal

A summary of the results from Examples 1 through 6 are below, along with whether or not each transformation had a vertical or horizontal effect on the graph.

Summary of Results from Examples 1 – 6
with notations about the vertical or horizontal effect on the graph, where
V = Vertical effect on graph H = Horizontal effect on graph

	First Set of Transformations **(with notations about horizontal/vertical effect)**	**Second Set of Transformations** **(with notations about horizontal/vertical effect)**	**Did (a) and (b) yield the same function?**
Ex 1	(a) Up 7 (V) Right 2 (H)	(b) Right 2 (H) Up 7 (V)	The functions were the same.
Ex 2	(a) Vertical stretch, factor of 2 (V) Down 5 (V)	(b) Down 5 (V) Vertical stretch, factor of 2 (V)	The functions were NOT the same.
Ex 3	(a) Reflect in *y*-axis (H) Up 6 (V)	(b) Up 6 (V) Reflect in *y*-axis (H)	The functions were the same.
Ex 4	(a) Reflect in *y*-axis (H) Left 2 (H)	(b) Left 2 (H) Reflect in *y*-axis (H)	The functions were NOT the same.
Ex 5	(a) Reflect in *x*-axis (V) Up 4 (V)	(b) Up 4 (V) Reflect in *x*-axis (V)	The functions were NOT the same.
Ex 6	(a) Horizontal shrink, factor of $\frac{1}{3}$ (H) Right 6 (H)	(b) Right 6 (H) Horizontal shrink, factor of $\frac{1}{3}$ (H)	The functions were NOT the same

Notice that in examples 1 and 3, the order of the transformations did not matter. In both of those examples, one of the transformations had a vertical effect on the graph, and the other transformation had a horizontal effect on the graph.

In examples 2, 4, 5 and 6, the order of the transformations did matter. Notice that example 2 had two vertically-oriented transformations, example 4 had two horizontally-oriented transformations, example 5 had two vertically-oriented transformations, and example 6 had two horizontally-oriented transformations.

When you perform two or more transformations that have a vertical effect on the graph, the order of those transformations may affect the final results. Similarly, when you perform two or more transformations that have a horizontal effect on the graph, the order of those transformations may affect the final results. The vertically-oriented transformations do not affect the horizontally-oriented transformations, and vice versa.

Let us now return to the function used at the start of this discussion:

Example Problem 7: Suppose that you want to graph $f(x)=3\sqrt{x+2}-7$. In what order can you perform the transformations to obtain the correct graph?

Solution:
First, decide on the transformations that need to be performed on $f(x)=3\sqrt{x+2}-7$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x)=\underset{\uparrow}{3}\sqrt{x+2}-7$	$f(x)=3\sqrt{x\underset{\uparrow}{+}2}-7$	$f(x)=3\sqrt{x+2}\underset{\uparrow}{-7}$
Vertical Stretch, factor of 3 (Vertical Effect)	Shift left 2 (Horizontal Effect)	Shift down 7 (Vertical Effect)

Notice that the shift to the left is the only transformation that has a horizontal effect on the graph. This transformation can be performed at any point in the graphing process.

We need to be more careful about the order in which we perform the vertical stretch and the downward shift, since they both have a vertical effect on the graph. Perform the following transformations algebraically on $g(x)=\sqrt{x}$ to see which one gives the desired function, $f(x)=3\sqrt{x+2}-7$. (The shift left is written first, but we could put that transformation at any point in the process and get the same result.)

Choice 1:
Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units:

$$g(x)=\sqrt{x} \quad \rightarrow \quad h(x)=\sqrt{x+2} \quad \rightarrow \quad k(x)=3\sqrt{x+2} \quad \rightarrow \quad f(x)=3\sqrt{x+2}-7$$

Left 2 | Vertical stretch, factor of 3 | Down 7

Choice 2:
Shift left 2 units, then shift downward 7 units, then stretch vertically by a factor of 3:

$$g(x)=\sqrt{x} \rightarrow h(x)=\sqrt{x+2} \rightarrow k(x)=\sqrt{x+2}-7 \rightarrow f(x)=3\left(\sqrt{x+2}-7\right)$$

Left 2 | Down 7 | Vertical stretch, factor of 3

Notice that our final result for Choice 2 can be written as $f(x)=3\sqrt{x+2}-21$, which is not the desired function.

We can see from the analysis above that Choice 1 yields the desired function for Example Problem 7, which means that in this particular example, the vertical stretch needs to be performed before the downward shift. Since the left shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units.
Stretch vertically by a factor of 3, then shift left 2 units, then shift downward 7 units.
Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units.

In this course, you would only need to give one of the answers from the above list (not all three). This explanation is given to help you understand that there can be multiple solutions for a given problem – and how to determine an acceptable order of transformations for the given problem.

Example Problem 8: Suppose that you want to graph $f(x)=\sqrt{-x+2}-7$. In what order can you perform the transformations to obtain the correct graph?

Solution:
First, decide on the transformations that need to be performed on $f(x)=\sqrt{-x+2}-7$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x)=\sqrt{-x+2}-7$	$f(x)=\sqrt{-x+2}-7$ ↑	$f(x)=\sqrt{-x+2}-7$ ↑
Reflect in the *y*-axis (Horizontal Effect)	Shift left 2 (Horizontal Effect)	Shift down 7 (Vertical Effect)

Notice that the downward shift is the only transformation that has a vertical effect on the graph. This transformation can be performed at any point in the graphing process.

We need to be more careful about the order in which we perform the reflection in the *y*-axis and the shift to the left, since they both have a horizontal effect on the graph. Perform the following transformations algebraically on $g(x)=\sqrt{x}$ to see which one gives the desired function, $f(x)=\sqrt{-x+2}-7$. (The downward shift is written first, but we could put that transformation at any point in the process and get the same result.)

Choice 1:
Downward shift of 7 units, then reflect in the *y*-axis, then shift left 2 units.

$$g(x)=\sqrt{x} \;\rightarrow\; h(x)=\sqrt{x}-7 \;\rightarrow\; k(x)=\sqrt{-x}-7 \;\rightarrow\; f(x)=\sqrt{-(x+2)}-7$$

Down 7   Reflect in the *y*-axis   Shift left 2 units

Notice that our final result for Choice 1 can be written as $f(x)=\sqrt{-x-2}-7$, which is not the desired function.

Choice 2:
Downward shift of 7 units, then shift left 2 units, then reflect in the *y*-axis.

$$g(x)=\sqrt{x} \;\rightarrow\; h(x)=\sqrt{x}-7 \;\rightarrow\; k(x)=\sqrt{x+2}-7 \;\rightarrow\; f(x)=\sqrt{-x+2}-7$$

Down 7   Shift left 2 units   Reflect in the *y*-axis

We can see from the analysis above that Choice 2 yields the desired function for Example Problem 8, which means that in this particular example, the shift to the left needs to be performed before the reflection in the *y*-axis. Since the downward shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.
Shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.
Shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units.

As mentioned in Example Problem 7, you would only need to give one of the answers from the above list (not all three). This explanation is given to help you understand that there can be multiple solutions for a given problem – and how to determine an acceptable order of transformations for the given problem.

Alternate Analysis of Example Problem 8:
It is possible to use a different set of transformations to yield the same result in Example Problem 8. Remember that the desired function is $f(x)=\sqrt{-x+2}-7$. You may notice that $f(x)=\sqrt{-x+2}-7=\sqrt{-(x-2)}-7$.

Looking at $f(x)=\sqrt{-(x-2)}-7$, notice that the $x+2$ from the original function is now $(x-2)$, which indicates a shift right of 2 units (rather than a shift left of 2 units).

The order of the downward shift again does not matter, but we need to decide if we should reflect in the *y*-axis and then shift right, or shift right and then reflect in the *y*-axis.

Choice 1:
Shift downward 7 units (can be done at any time), reflect in the *y*-axis, then shift right 2 units:

$$g(x)=\sqrt{x} \;\rightarrow\; h(x)=\sqrt{x}-7 \;\rightarrow\; k(x)=\sqrt{-x}-7 \;\rightarrow\; f(x)=\sqrt{-(x-2)}-7$$

Down 7 | Reflect in the *y*-axis | Shift right 2 units

Notice that our final result for Choice 1 can be written as $f(x)=\sqrt{-x+2}-7$, which is the desired function.

Choice 2:
Shift downward 7 units (can be done at any time), shift right 2 units, then reflect in the *y*-axis.

$$g(x)=\sqrt{x} \;\rightarrow\; h(x)=\sqrt{x}-7 \;\rightarrow\; k(x)=\sqrt{x-2}-7 \;\rightarrow\; f(x)=\sqrt{-x-2}-7$$

Down 7 | Shift right 2 units | Reflect in the *y*-axis

Notice that Choice 2 does not yield the desired function.

We can see from the alternate analysis above that Choice 1 yields the desired function for Example Problem 8, which means that in this particular example, the reflection in the y-axis needs to be performed before the shift to the right. Since the downward shift can be performed at any point in the process, any of the following order of transformations would yield the correct graph:

Shift downward 7 units, then reflect in the *y*-axis, then shift right 2 units.
Reflect in the *y*-axis, then shift downward 7 units, then shift right 2 units.
Reflect in the *y*-axis, then shift right 2 units, then shift downward 7 units.

Remember that when we used a shift to the left instead, we obtained the following solutions as well:

Shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.
Shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.
Shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units.

In all of the examples above, we have discussed problems where there were two or three transformations. **What if there are four or more transformations?** In that case, look at the vertically-oriented transformations and decide the order in which they need to be done. Look at the horizontally-oriented transformations and decide the order in which they need to be done. Then remember that the vertically-oriented transformations have no effect on the horizontally-oriented transformations, and vice versa.

Example Problem 9: Suppose that you want to graph $f(x) = 3\sqrt{-x+2} - 7$.

(a) In what order can you perform the transformations to obtain the correct graph?
(b) Graph the function.

Solution:

(a) In order to save time, we have chosen an example that is similar to Example Problems 7 and 8.

We have the following transformations, not necessarily to be performed in this order:

Stretch vertically, factor of 3	Vertically-oriented transformation
Reflection in the *y*-axis	Horizontally-oriented transformation
Shift left 2 units*	Horizontally-oriented transformation
Shift downward 7 units	Vertically-oriented transformation

*For simplicity, we are choosing to focus on a left shift for this problem, instead of factoring out a negative under the radical, where $f(x) = 3\sqrt{-(x-2)} - 7$, and focusing on a right shift instead. As seen in Example Problem 8, we would obtain additional solutions if we considered a right shift as an alternate means of solving this problem. (The point of these detailed explanations is not for the student to be able to list all possible orders of transformations, but to be able to determine one order of transformations for any given problem which would yield a correct graph.)

Let us look first at the vertically-oriented transformations. (Temporarily put aside the horizontally-oriented transformations, and just look at what is happening 'outside' the radical sign.) As in Example Problem 7, the vertical stretch needs to be performed before the downward shift.

Next look at the horizontally-oriented transformations. (Temporarily put aside the vertically-oriented transformations, and just look at what is happening under the radical sign.) As in Example Problem 8, the shift to the left needs to be performed before the reflection in the *y*-axis.

Remember that the vertically-oriented transformations do not affect the horizontally-oriented transformations, and vice versa. There are many correct solutions to this problem. Just be sure that in your answer, the vertical stretch is performed before the downward shift, and the shift to the left is performed before the reflection in the *y*-axis.

All the solutions involving a left shift are shown below. **You do not need to find all of the answers below; any one of the solutions below would be acceptable.** (Note: If we looked at all the possibilities involving a right shift, we would add six more solutions which would yield the same graph.)

Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis.

Shift left 2 units, then reflect in the *y*-axis, then stretch vertically by a factor of 3, then shift downward 7 units.

Stretch vertically by a factor of 3, then shift left 2 units, then shift downward 7 units, then reflect in the *y*-axis.

Stretch vertically by a factor of 3, then shift left 2 units, then reflect in the *y*-axis, then shift downward 7 units,.

Shift left 2 units, then stretch vertically by a factor of 3, then reflect in the *y*-axis, then shift downward 7 units.

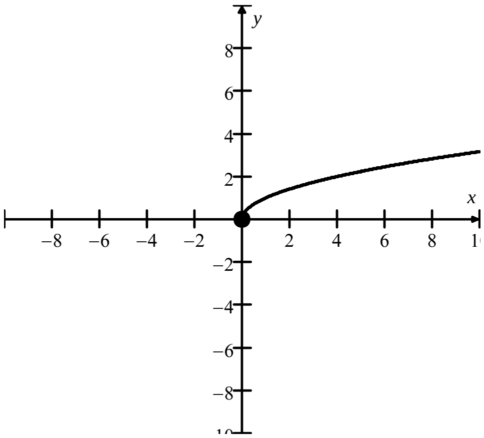
Shift left 2 units, then stretch vertically by a factor of 3, then shift downward 7 units, then reflect in the *y*-axis.

(b) To graph the function, we can choose any one of the solutions shown above (and obtain the same result). We will perform the transformations in the order listed for the first solution listed above:

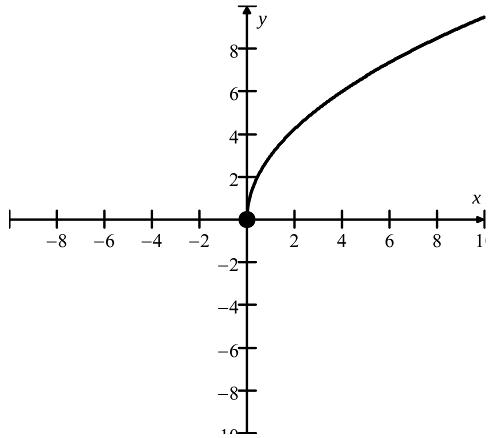
Starting with the base graph $y = \sqrt{x}$:

Stretch vertically by a factor of 3, then shift downward 7 units, then shift left 2 units, then reflect in the *y*-axis. All steps are shown consecutively below.

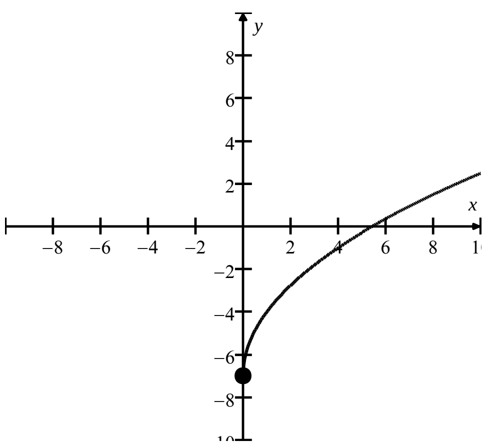
Step 1: $y=\sqrt{x}$



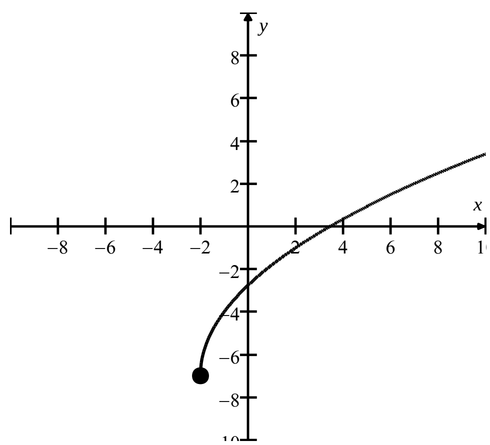
Step 2: $y=3\sqrt{x}$
(Stretch vertically by a factor of 3)



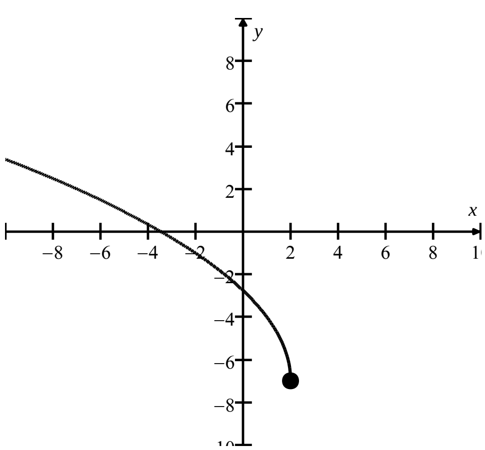
Step 3: $y=3\sqrt{x}-7$
(Shift downward 7 units)



Step 4: $y=3\sqrt{x+2}-7$
(Shift left 2 units)



Step 5: $y=f\left(x\right)=3\sqrt{-x+2}-7$
(Reflect in the *y*-axis)



Step 5 represents the final result.

Additional Example 1:

Sketch the graph of the function $f(x) = -2|x| + 1$. Do not plot points, but instead apply transformations to the graph of a standard function.

Analysis of the Order of Transformations for Additional Example 1:
In order to graph this problem correctly, we need to choose a correct order of transformations. First, decide on the transformations that need to be performed on $f(x) = -2|x| + 1$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x) = \underset{\uparrow}{-}2\|x\| + 1$	$f(x) = -\underset{\uparrow}{2}\|x\| + 1$	$f(x) = -2\|x\| \underset{\uparrow}{+1}$
Reflect in the *x*-axis (Vertical Effect)	Stretch vertically, factor of 2 (Vertical Effect)	Shift up 1 (Vertical Effect)

Remember that any vertically-oriented transformations have no effect on horizontally-oriented transformations, and vice versa. However, the order in which you perform vertically-oriented transformations may make a difference in the graph, and the order in which you perform horizontally-oriented transformations may make a difference in the graph. In this example, notice that all three transformations have a vertical effect on the graph, so the order in which we choose the transformations may matter.

You do NOT need to evaluate all the possible combinations in which the above transformations can be performed, but <u>you need to find one</u> which algebraically yields the correct function (and therefore also yields the correct graph). For the purpose of being complete in our solution, we have shown all possible combinations of transformations below so that you can evaluate whether or not your order of transformations is correct.

Choice 1: Reflect in the *x*-axis, stretch vertically by a factor of 2, shift up 1
$g(x) = |x| \rightarrow h(x) = -|x| \rightarrow k(x) = -2|x| \rightarrow f(x) - 2|x| + 1$
Choice 1 yields the desired function, $f(x) = -2|x| + 1$.

Choice 2: Reflect in the *x*-axis, shift up 1, stretch vertically by a factor of 2.
$g(x) = |x| \rightarrow h(x) = -|x| \rightarrow k(x) = -|x| + 1 \rightarrow f(x) = 2(-|x| + 1) = -2|x| + 2$

Choice 2 does NOT yield the desired function, $f(x) = -2|x| + 1$.

Choice 3: Stretch vertically by a factor of 2, reflect in the *x*-axis, shift up 1
$g(x) = |x| \rightarrow h(x) = 2|x| \rightarrow k(x) = -2|x| \rightarrow f(x) = -2|x| + 1$
Choice 3 yields the desired function, $f(x) = -2|x| + 1$.

Choice 4: Stretch vertically by a factor of 2, shift up 1, reflect in the *x*-axis

$g(x)=|x| \rightarrow h(x)=2|x| \rightarrow k(x)=2|x|+1 \rightarrow f(x)=-(2|x|+1)=-2|x|-1$

Choice 4 does NOT yield the desired function, $f(x)=-2|x|+1$.

Choice 5:
Shift up 1, reflect in the *x*-axis, stretch vertically by a factor of 2.

$g(x)=|x| \rightarrow h(x)=|x|+1 \rightarrow k(x)=-(|x|+1) \rightarrow f(x)=-2(|x|+1)=-2|x|-2$

Choice 5 does NOT yield the desired function, $f(x)=-2|x|+1$.

Choice 6:
Shift up 1, stretch vertically by a factor of 2, reflect in the *x*-axis.

$g(x)=|x| \rightarrow h(x)=|x|+1 \rightarrow k(x)=2(|x|+1) \rightarrow f(x)=-2(|x|+1)=-2|x|-2$

Choice 6 does NOT yield the desired function, $f(x)=-2|x|+1$.

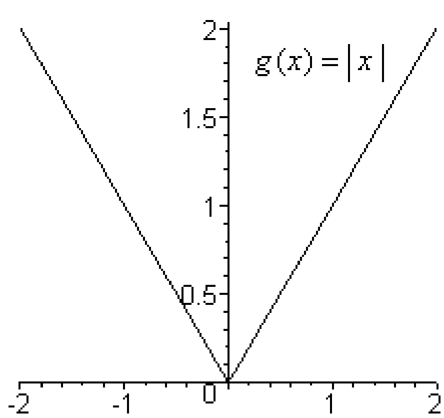
We can see from the six choices above that the following combinations of transformations yield the desired function (and that all the rest will yield an incorrect result):

Choice 1: Reflect in the *x*-axis, stretch vertically by a factor of 2, shift up 1
Choice 3: Stretch vertically by a factor of 2, reflect in the *x*-axis, shift up 1

If you were working this problem, you could stop analyzing choices as soon as you find one that works – and then proceed to graph the function. The solution shown below uses the order of transformations shown in Choice 3.

Solution to Additional Example 1:

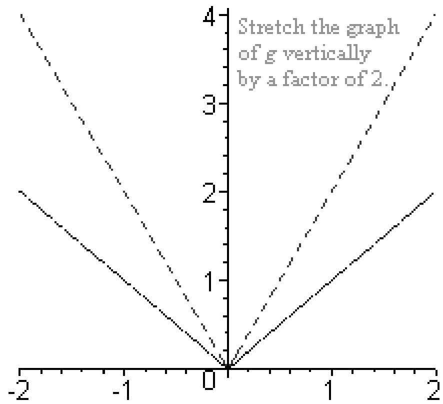
Begin with the graph of $g(x)=|x|$ shown below.



First stretch the graph of g vertically by a factor of 2.

$$\left[f(x) = -\boxed{2}\,|x| + 1 \right]$$

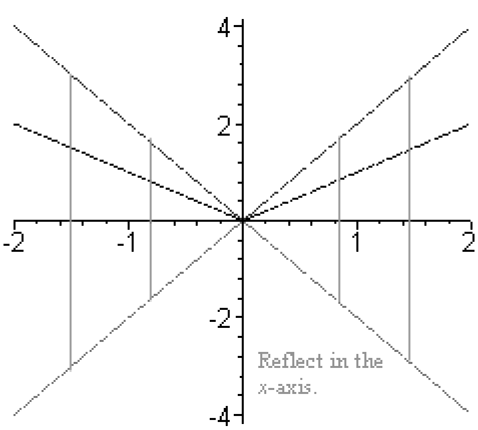
Stretch by a factor of 2.



Next reflect in the x-axis.

$$\left[f(x) = \boxed{-}2\,|x| + 1 \right]$$

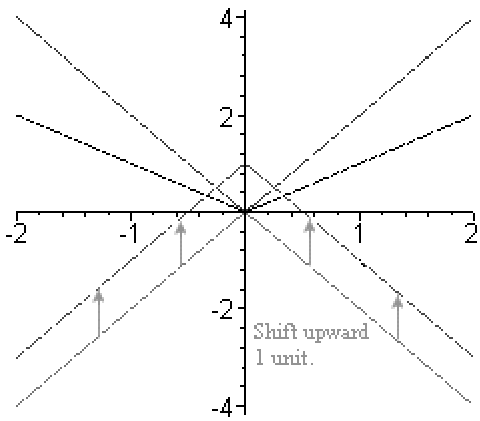
Reflect in the x-axis.



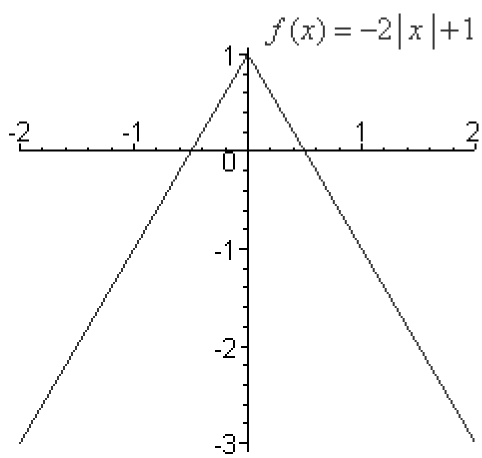
Now shift upward
1 unit.

$\left[f(x) = -2|x| + 1 \right]$

Shift upward
1 unit.



The graph of the given function is shown below.



Additional Example 2:

Sketch the graph of the function $f(x) = -|x-3| - 2$. Do not plot points, but instead apply transformations to the graph of a standard function.

Analysis of the Order of Transformations for Additional Example 2:

In order to graph this problem correctly, we need to choose a correct order of transformations. First, decide on the transformations that need to be performed on $f(x) = -|x-3| - 2$ (without consideration of correct order). Make a note of whether each transformation has a horizontal or vertical effect on the graph.

$f(x) = \underset{\uparrow}{-}|x-3| - 2$	$f(x) = -|x\underset{\uparrow}{-}3| - 2$	$f(x) = -|x-3|\underset{\uparrow}{-2}$
Reflect in the *x*-axis	Shift right 3	Shift down 2
(Vertical Effect)	(Horizontal Effect)	(Vertical Effect)

Remember that any vertically-oriented transformations have no effect on horizontally-oriented transformations, and vice versa. However, the order in which you perform vertically-oriented transformations may make a difference in the graph, and the order in which you perform horizontally-oriented transformations may make a difference in the graph.

Notice that the shift to the right is the only transformation that has a horizontal effect on the graph. Since it has no effect on the vertically-oriented transformations, the right shift can be performed at any point in the graphing process.

We need to be more careful about the order in which we perform the reflection in the *x*-axis and the downward shift, since they both have a vertical effect on the graph.

Remember that you do NOT need to evaluate all the possible combinations in which the above transformations can be performed, but <u>you need to find one</u> which algebraically yields the correct function (and therefore also yields the correct graph).

Perform the following transformations algebraically on $g(x) = |x|$ to see which one gives the desired function, $f(x) = -|x-3| - 2$. (In the choices below, the shift to the right is written first, but we could put that transformation at any point in the process and get the same result.)

Choice 1: Shift right 3, reflect in the *x*-axis, shift down 2

$g(x) = |x| \to h(x) = |x-3| \to k(x) = -|x-3| \to f(x) - |x-3| - 2$

Choice 1 yields the desired function, $f(x) - |x-3| - 2$.

Choice 2: Shift right 3, shift down 2, reflect in the *x*-axis.

$g(x)=|x| \;\rightarrow\; h(x)=|x-3| \;\rightarrow\; k(x)=|x-3|-2 \;\rightarrow\; f(x)=-(|x-3|-2)$

Notice that in Choice 2, $f(x)=-(|x-3|-2)=-|x-3|+2$, and does NOT yield the desired function, $f(x)=-|x-3|-2$.

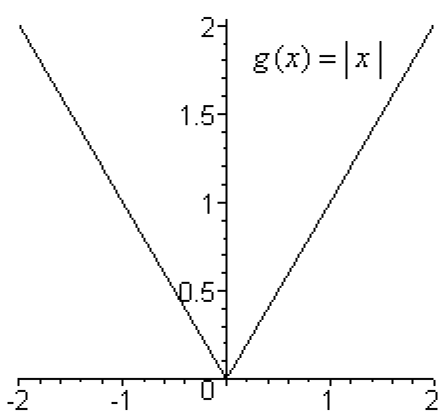
We can see from the choices above that the reflection in the *x*-axis needs to be performed before the downward shift. Since the right shift is the only horizontally-oriented transformation, it can be performed at any point in the process. Therefore, any of the following combinations of transformations would yield the correct graph:

Shift right 3, reflect in the *x*-axis, shift down 2
Reflect in the *x*-axis, shift right 3, shift down 2
Reflect in the *x*-axis, shift down 2, shift right 3

The solution shown below uses the order of transformations shown in Choice 1.

Solution to Additional Example 2:

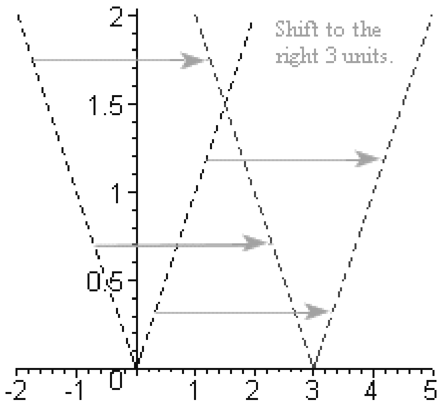
Begin with the graph of $g(x)=|x|$ shown below.



First shift the
graph of g 3 units
to the right.

$\left[f(x) = -|x - 3| - 2\right]$

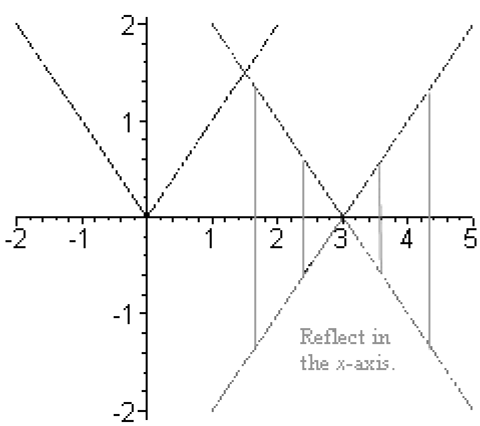
Shift to the
right 3 units.



Next reflect in the
x-axis.

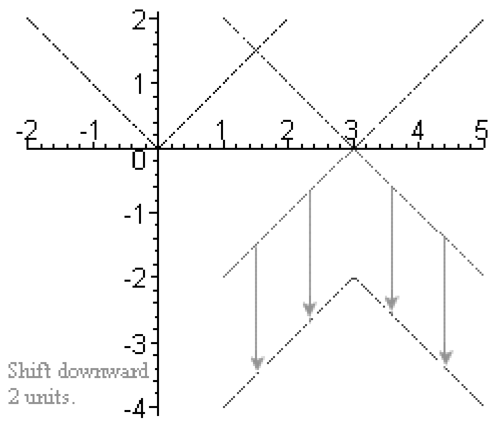
$\left[f(x) = -|x - 3| - 2\right]$

Reflect in
the x-axis.

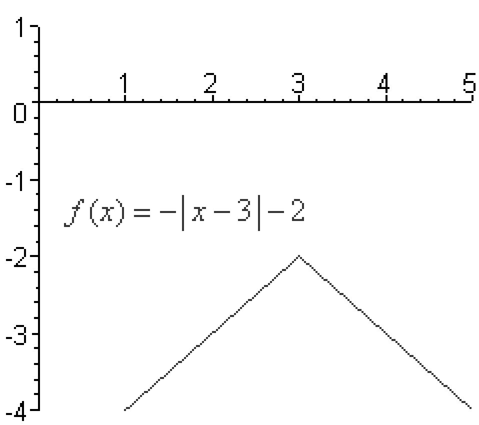


Now shift downward
2 units.
$\left[f(x) = -|x-3|-2\right]$

Shift downward
2 units.



The graph of the given function is shown below.



Exercise Set 1.3: Transformations of Graphs

Suppose that a student looks at a transformation of $y = f(x)$ and breaks it into the following steps. State the transformation that occurs in each step below.

1. $y = 2f(-x-3)+4$

(a) From: $y = f(x)$ To: $y = f(x-3)$

(b) From: $y = f(x-3)$ To: $y = f(-x-3)$

(c) From: $y = f(-x-3)$ To: $y = 2f(-x-3)$

(d) From: $y = 2f(-x-3)$
To: $y = 2f(-x-3)+4$

2. $y = -\frac{1}{4}f(1-x)-5$

First notice that $(1-x)$ *is equivalent to* $(-x+1)$

(a) From: $y = f(x)$
To: $y = f(x+1)$

(b) From: $y = f(x+1)$
To: $y = f(-x+1) = f(1-x)$

(c) From: $y = f(1-x)$
To: $y = \frac{1}{4}f(1-x)$

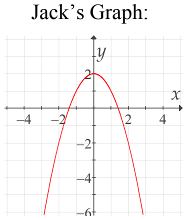
(d) From: $y = \frac{1}{4}f(1-x)$
To: $y = -\frac{1}{4}f(1-x)$

(e) From: $y = -\frac{1}{4}f(1-x)$
To: $y = -\frac{1}{4}f(1-x)-5$

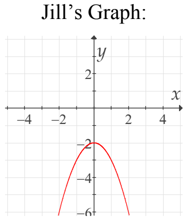
Answer the following.

3. Jack and Jill are graphing the function $f(x) = 2 - x^2$. Starting with the graph of $y = x^2$, Jack first reflects the graph in the x-axis and then shifts upward two units. Jill, on the other hand, first shifts the graph $y = x^2$ upward two units and then reflects in the x-axis. Their graphs are shown below.

Jack's Graph:



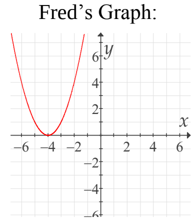
Jill's Graph:



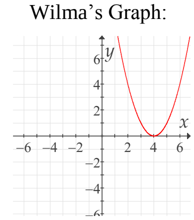
(a) Who is correct, Jack or Jill?
(b) Analyze the two methods and explain the algebraic difference between the two. Use this analysis to justify your answer in part (a).

4. Fred and Wilma are graphing the function $f(x) = (-x+4)^2$. Starting with the graph of $y = x^2$, Fred first reflects the graph in the y-axis and then shifts four units to the left. Wilma, on the other hand, first shifts the graph $y = x^2$ four units to the left and then reflects in the y-axis. Their graphs are shown below.

Fred's Graph:



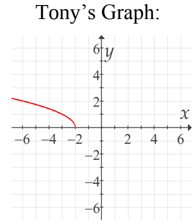
Wilma's Graph:



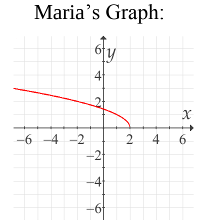
(a) Who is correct, Fred or Wilma?
(b) Analyze the two methods and explain the algebraic difference between the two. Use this analysis to justify your answer in part (a).

5. Tony and Maria are graphing the function $f(x) = \sqrt{-2-x}$. Starting with the graph of $y = \sqrt{x}$, Tony first shifts the graph two units to the right and then reflects in the y-axis. Maria, on the other hand, first reflects the graph $y = \sqrt{x}$ in the y-axis and then shifts two units to the right. Their graphs are shown below.

Tony's Graph:



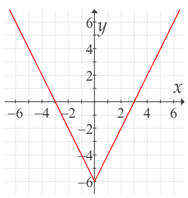
Maria's Graph:



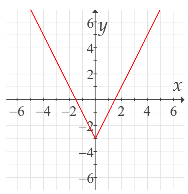
(a) Who is correct, Tony or Maria?
(b) Analyze the two methods and explain the algebraic difference between the two. Use this analysis to justify your answer in part (a).

6. Bart and Lisa are graphing the function $f(x) = 2|x| - 3$. Starting with the graph of $y = |x|$, Bart first shifts the graph downward three units and then stretches the graph vertically by a factor of 2. Lisa, on the other hand, first stretches the graph $y = |x|$ vertically by a factor of 2 and then shifts the graph downward three units.

Bart's Graph:



Lisa's Graph:



(a) Who is correct, Bart or Lisa?
(b) Analyze the two methods and explain the algebraic difference between the two. Use this analysis to justify your answer in part (a).

Matching. The left-hand column contains equations that represent transformations of $f(x) = x^2$. Match the equations on the left with the description on the right of how to obtain the graph of g from the graph of f.

7. $g(x) = (x-4)^2$

8. $g(x) = x^2 - 4$

9. $g(x) = x^2 + 4$

10. $g(x) = (x+4)^2$

11. $g(x) = -x^2$

12. $g(x) = (-x)^2$

13. $g(x) = 4x^2$

14. $g(x) = \frac{1}{4}x^2$

15. $g(x) = -x^2 - 4$

16. $g(x) = (x+4)^2 + 3$

17. $g(x) = -(x-3)^2 + 4$

18. $g(x) = (-x+4)^2$

A. Reflect in the *x*-axis.

B. Shift left 4 units, then reflect in the *y*-axis.

C. Reflect in the *x*-axis, then shift downward 4 units.

D. Shift right 4 units.

E. Shift right 3 units, then reflect in the *x*-axis, then shift upward 4 units.

F. Shift upward 4 units.

G. Reflect in the *y*-axis.

H. Shift left 4 units, then shift upward 3 units.

I. Shift left 4 units.

J. Shift downward 4 units.

K. Stretch vertically by a factor of 4.

L. Shrink vertically by a factor of $\frac{1}{4}$.

Write the equation that results when the following transformations are applied to the given standard function. Then state if any of the resulting functions in (a)-(e) are equivalent.

19. Standard function: $y = x^3$

(a) Shift right 7 units, then reflect in the *x*-axis, then stretch vertically by a factor of 5, then shift upward 1 unit.

(b) Reflect in the *x*-axis, then shift right 7 units, then stretch vertically by a factor of 5, then shift upward 1 unit.

(c) Stretch vertically by a factor of 5, then shift upward 1 unit, then shift right 7 units, then reflect in the *x*-axis.

(d) Shift right 7 units, then shift upward 1 unit, then reflect in the *x*-axis, then stretch vertically by a factor of 5.

(e) Reflect in the *x*-axis, then shift left 7 units, then stretch vertically by a factor of 5, then shift upward 1 unit.

(f) Which, if any, of the resulting functions in (a)-(e) are equivalent?

20. Standard function: $y = \sqrt{x}$

(a) Reflect in the *y*-axis, then shift left 2 units, then shift downward 4 units, then reflect in the *x*-axis

(b) Shift left 2 units, then reflect in the *y*-axis, then reflect in the *x*-axis, then shift downward 4 units.

(c) Reflect in the *y*-axis, then reflect in the *x*-axis, then shift downward 4 units, then shift right 2 units.

(d) Reflect in the *x*-axis, then shift left 2 units, then shift downward 4 units, then reflect in the *y*-axis.

(e) Shift downward 4 units, then shift left 2 units, then reflect in the *y*-axis, then reflect in the *x*-axis.

(f) Which, if any, of the resulting functions in (a)-(e) are equivalent?

Continued on the next page…

Exercise Set 1.3: Transformations of Graphs

Describe how the graph of *g* is obtained from the graph of *f*. (Do not sketch the graph.)

21. $f(x)=\sqrt{x}$, $\quad g(x)=\sqrt{-x}\ -2$

22. $f(x)=x^3$, $\quad g(x)=-2(x+5)^3$

23. $f(x)=|x|$, $\quad g(x)=-5|x-2|+1$

24. $f(x)=x^2$, $\quad g(x)=\frac{1}{6}(x+3)^2-7$

25. $f(x)=\dfrac{1}{x}$, $\quad g(x)=\dfrac{3}{x+8}+2$

26. $f(x)=\sqrt[3]{x}$, $\quad g(x)=\sqrt[3]{-x}+4$

Describe how the graphs of each of the following functions can be obtained from the graph of *y = f(x)*.

27. $y=f(x)+1$

28. $y=f(x-7)$

29. $y=f(-x)+3$

30. $y=-f(x+3)-8$

31. $y=-\frac{1}{4}f(x-2)-5$

32. $y=-5f(-x)+1$

33. $y=f(7-x)+2$

34. $y=f(-x-5)-7$

Sketch the graph of each of the following functions. Do not plot points, but instead apply transformations to the graph of a standard function.

35. $f(x)=x-3$

36. $f(x)=5-x$

37. $f(x)=-3x+1$

38. $f(x)=2x-7$

39. $f(x)=x^2+3$

40. $f(x)=(x-5)^2$

41. $f(x)=6-x^2$

42. $f(x)=2-(x-1)^2$

43. $f(x)=-3(x-4)^2-2$

44. $f(x)=(x+5)^2+3$

45. $f(x)=6-\sqrt{x+2}$

46. $f(x)=\frac{1}{2}\sqrt{-x}+1$

47. $f(x)=\sqrt{-x+4}+2$

48. $f(x)=\sqrt{5-x}-1$

49. $f(x)=2|x+5|-3$

50. $f(x)=-|x-2|+4$

51. $f(x)=-(x-4)^3+1$

52. $f(x)=-x^3-5$

53. $f(x)=\dfrac{1}{x-3}+6$

54. $f(x)=-\dfrac{2}{x+4}$

55. $f(x)=-\dfrac{4}{x}+3$

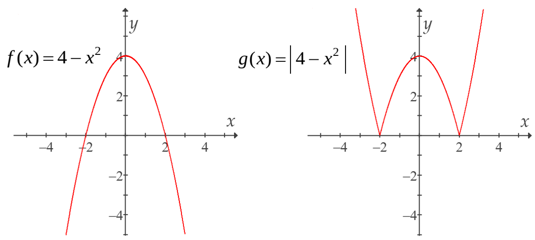
56. $f(x)=\sqrt[3]{x-6}$

57. $f(x)=\sqrt[3]{-x}+2$

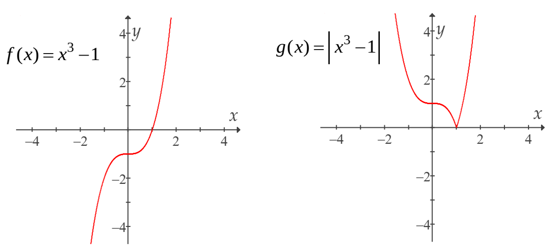
58. $f(x)=-\sqrt[3]{x+1}-5$

Answer the following.

59. The graphs of $f(x) = 4 - x^2$ and $g(x) = \left|4 - x^2\right|$ are shown below. Describe how the graph of g was obtained from the graph of f.



60. The graphs of $f(x) = x^3 - 1$ and $g(x) = \left|x^3 - 1\right|$ are shown below. Describe how the graph of g was obtained from the graph of f.



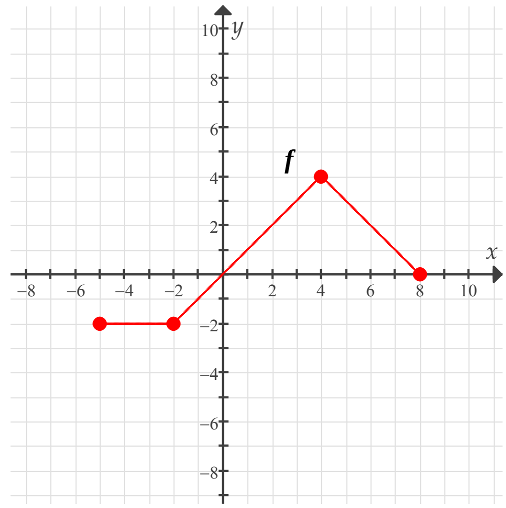
Sketch the graphs of the following functions:

61. **(a)** $f(x) = x^2 - 9$ **(b)** $g(x) = \left|x^2 - 9\right|$

62. **(a)** $f(x) = \dfrac{1}{x}$ **(b)** $g(x) = \left|\dfrac{1}{x}\right|$

Continued in the next column…

The graph of $y = f(x)$ is given below. Sketch the graph of each of the following functions.



63. $y = f(x+2)$

64. $y = f(x) - 3$

65. $y = f(x-2) - 1$

66. $y = f(x+1) + 5$

67. $y = f(-x)$

68. $y = -f(x)$

69. $y = 2f(x)$

70. $y = \frac{1}{2}f(x)$

71. $y = -2f(x+1)$

72. $y = f(-x) - 4$