Vibrations of Single Degree of Freedom Systems

CEE 201L. Uncertainty, Design, and Optimization
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This document describes free and forced dynamic responses of single degree of freedom (SDOF) systems. The prototype single degree of freedom system is a spring-mass-damper system in which the spring has no damping or mass, the mass has no stiffness or damping, the damper has no stiffness or mass. Furthermore, the mass is allowed to move in only one direction. The horizontal vibrations of a single-story building can be conveniently modeled as a single degree of freedom system. Part 1 of this document describes some useful trigonometric identities. Part 2 shows how damped SDOF systems vibrate freely after being released from an initial displacement with some initial velocity. Part 3 covers the resposne of damped SDOF systems to persistent sinusoidal forcing.

Consider the structural system shown in Figure 1, where:

$f(t)$ = external excitation force
$x(t)$ = displacement of the center of mass of the moving object
m = mass of the moving object, $f_I = \frac{d}{dt}(m\dot{x}(t)) = m\ddot{x}(t)$
c = linear viscous damping coefficient, $f_D = c\dot{x}(t)$
k = linear elastic stiffness coefficient, $f_S = kx(t)$

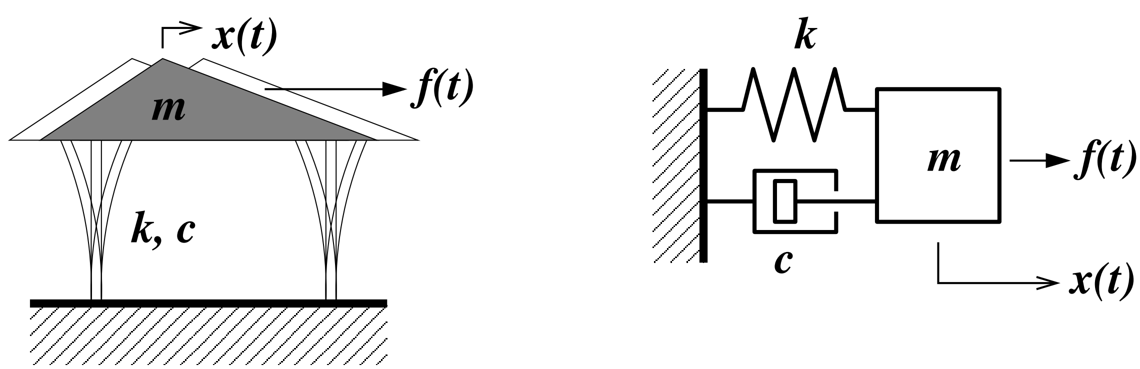


Figure 1. The proto-typical single degree of freedom oscillator.

The kinetic energy $T(x, \dot{x})$, the potential energy, $V(x)$, and the external forcing and dissipative forces, $p(x, \dot{x})$, are

$$T(x, \dot{x}) = \frac{1}{2} m (\dot{x}(t))^2 \quad (1)$$

$$V(x) = \frac{1}{2} k (x(t))^2 \quad (2)$$

$$p(x, \dot{x}) = -c\dot{x}(t) + f(t) \quad (3)$$

The general form of the differential equation describing a SDOF oscillator follows directly from Lagrange's equation,

$$\frac{d}{dt}\frac{\partial T(x, \dot{x})}{\partial \dot{x}} - \frac{\partial T(x, \dot{x})}{\partial x} + \frac{\partial V(x)}{\partial x} - p(x, \dot{x}) = 0 \ , \quad (4)$$

or from simply balancing the forces on the mass,

$$\sum F = 0 : f_I + f_D + f_S = f(t) \ . \quad (5)$$

Either way, the equation of motion is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t), \quad x(0) = d_o, \quad \dot{x}(0) = v_o \quad (6)$$

where the initial displacement is d_o, and the initial velocity is v_o.

The solution to equation (6) is the sum of a homogeneous part (free response) and a particular part (forced response). This document describes free responses of all types and forced responses to simple-harmonic forcing.

1 Trigonometric and Complex Exponential Expressions for Oscillations

1.1 Constant Amplitude

An oscillation, $x(t)$, with amplitude $\bar{X}$ and frequency ω can be described by sinusoidal functions. These sinusoidal functions may be equivalently written in terms of complex exponentials $e^{\pm i\omega t}$ with complex coefficients, $X = A + iB$ and $X^* = A - iB$. (The complex constant X^* is called the *complex conjugate* of X.)

$$x(t) = \bar{X} \cos(\omega t + \theta) \quad (7)$$

$$= a \cos(\omega t) + b \sin(\omega t) \quad (8)$$

$$= X \ e^{+i\omega t} + X^* \ e^{-i\omega t} \quad (9)$$

To relate equations (7) and (8), recall the cosine of a sum of angles,

$$\bar{X}\cos(\omega t+\theta) = \bar{X}\cos(\theta)\cos(\omega t) - \bar{X}\sin(\theta)\sin(\omega t) \tag{10}$$

Comparing equations (10) and (8), we see that

$$a = \bar{X}\cos(\theta)\ , \qquad b = -\bar{X}\sin(\theta)\ , \qquad \text{and} \qquad a^2+b^2 = \bar{X}^2\ . \tag{11}$$

Also, the ratio b/a provides an equation for the phase shift, θ,

$$\tan(\theta) = -\frac{b}{a} \tag{12}$$

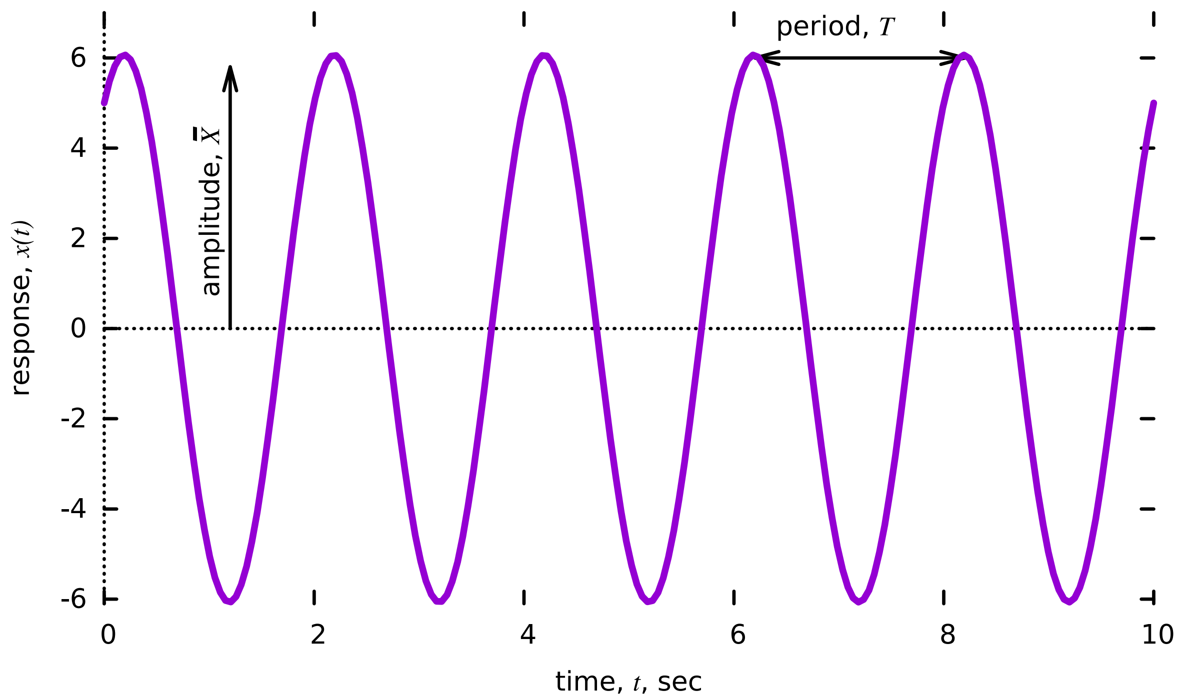


Figure 2. A constant-amplitude oscillation.

To relate equations (8) and (9), recall the expression for a complex exponent in terms of sines and cosines,

$$\begin{aligned} X\ e^{+i\omega t} + X^*\ e^{-i\omega t} &= (A+iB)\ (\cos(\omega t)+i\sin(\omega t)) + \\ &\quad (A-iB)\ (\cos(\omega t)-i\sin(\omega t)) \qquad (13) \\ &= A\cos(\omega t) - B\sin(\omega t) + iA\sin(\omega t) + iB\cos(\omega t) + \\ &\quad A\cos(\omega t) - B\sin(\omega t) - iA\sin(\omega t) - iB\cos(\omega t) \\ &= 2A\cos(\omega t) - 2B\sin(\omega t) \qquad (14) \end{aligned}$$

Comparing equations (14) and (8), we see that

$$a = 2A \ , \qquad b = -2B \ , \quad \text{and} \quad \tan(\theta) = \frac{B}{A} \ . \tag{15}$$

Any sinusoidal oscillation $x(t)$ can be expressed equivalently in terms of equations (7), (8), or (9); the choice depends on the application, and the problem to be solved. Equations (7) and (8) are easier to interpret as describing a sinusoidal oscillation, but equation (9) can be much easier to work with mathematically. These notes make use of all three forms.

One way to interpret the compex exponential notation is as the sum of complex conjugates,

$$Xe^{+i\omega t} = [A\cos(\omega t) - B\sin(\omega t)] + i[A\sin(\omega t) + B\cos(\omega t)]$$

and

$$X^* e^{-i\omega t} = [A\cos(\omega t) - B\sin(\omega t)] - i[A\sin(\omega t) + B\cos(\omega t)]$$

as shown in Figure 3. The sum of complex conjugate pairs is real, since the imaginary parts cancel out.

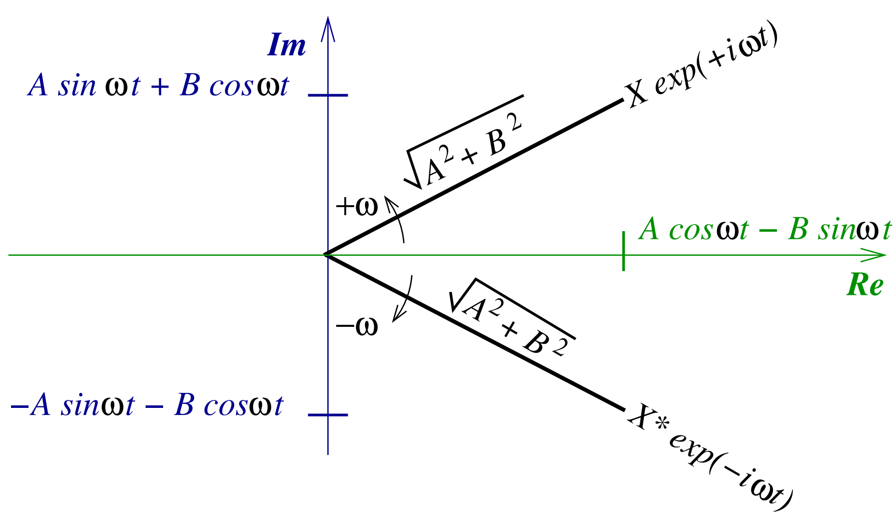


Figure 3. Complex conjugate oscillations.

The amplitude, $\bar{X}$, of the oscillation $x(t)$ can be found by finding the the sum of the complex amplitudes $|X|$ and $|X^*|$.

$$\bar{X} = |X| + |X^*| \ = \ 2\sqrt{A^2 + B^2} = \sqrt{a^2 + b^2} \tag{16}$$

Note, again, that equations (7), (8), and (9) are all equivalent using the relations among (a, b), (A, B), $\bar{X}$, and θ given in equations (11), (12), (15), and (16).

1.2 Decaying Amplitude

The dynamic response of damped systems decays over time. Note that damping may be introduced into a structure through diverse mechanisms, including linear viscous damping, nonlinear viscous damping, visco-elastic damping, friction damping, and plastic deformation. ll but linear viscous damping are somewhat complicated to analyze, so we will restrict our attention to linear viscous damping, in which the damping force f_D is proportional to the velocity, $f_D = c\dot{x}$.

To describe an oscillation which decays with time, we can multiply the expression for a constant amplitude oscillation by a positive-valued function which decays with time. Here we will use a real exponential, $e^{\sigma t}$, where $\sigma < 0$.

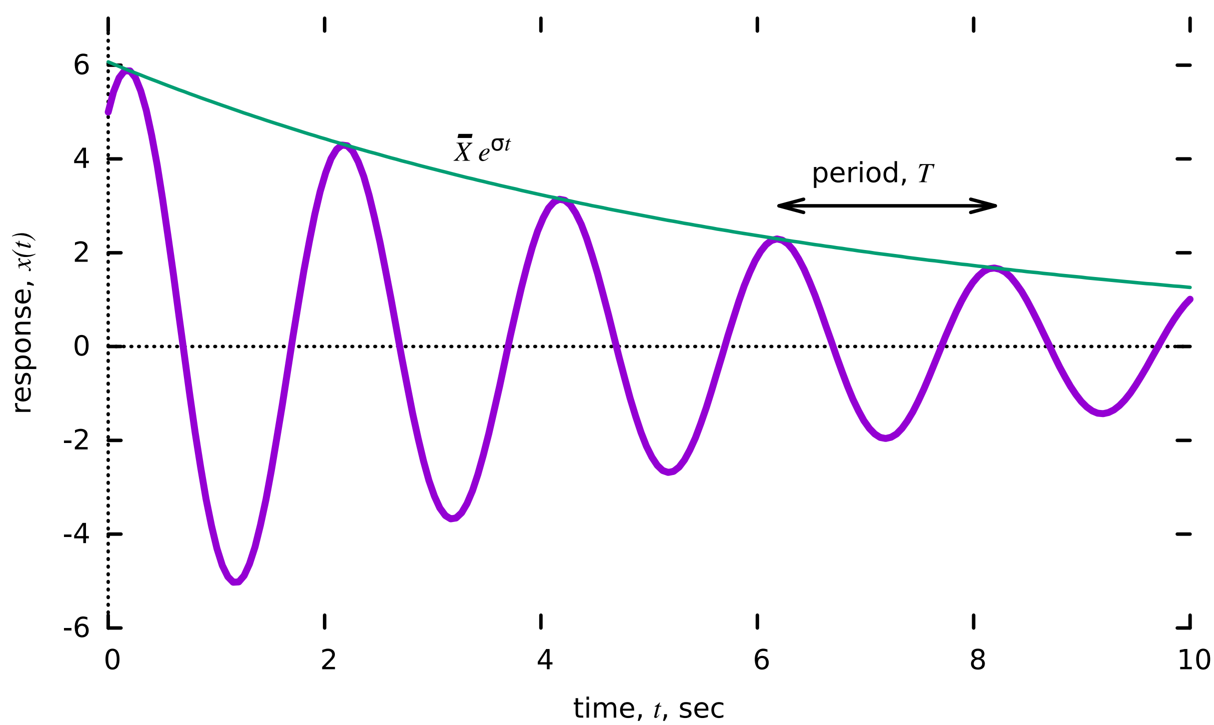


Figure 4. A decaying oscillation.

Multiplying equations (7) through (9) by $e^{\sigma t}$,

$$\begin{aligned} x(t) &= e^{\sigma t}\ \bar{X}\ (\cos(\omega t + \theta)) && (17)\\ &= e^{\sigma t}\ (a\cos(\omega t) + b\sin(\omega t)) && (18)\\ &= e^{\sigma t}\ (Xe^{i\omega t} + X^* e^{-i\omega t}) && (19)\\ &= Xe^{(\sigma+i\omega)t} + X^* e^{(\sigma-i\omega)t} && (20)\\ &= Xe^{\lambda t} + X^* e^{\lambda^* t} && (21) \end{aligned}$$

Again, note that *all* of the above equations are *exactly* equivalent. The exponent λ is complex, $\lambda = \sigma + i\omega$ and $\lambda^* = \sigma - i\omega$. If σ is negative, then these equations describe an oscillation with exponentially decreasing amplitudes. Note that in equation (18) the unknown constants are σ, ω, a, and b. Angular frequencies, ω, have units of radians per second. Circular frequencies, $f = \omega/(2\pi)$ have units of cycles per second, or Hertz. Periods, $T = 2\pi/\omega$, have units of seconds.

In the next section we will find that for an un-forced vibration, σ and ω are determined from the mass, damping, and stiffness of the system. We will see that the constant a equals the initial displacement d_o, but that the constant b depends on the initial displacement and velocity, as well mass, damping, and stiffness.

2 Free response of systems with mass, stiffness and damping

Using equation (21) to describe the free response of a single degree of freedom system, we will set $f(t) = 0$ and will substitute $x(t) = Xe^{\lambda t}$ into equation (6).

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \ , \quad x(0) = d_o \ , \quad \dot{x}(0) = v_o \ , \tag{22}$$

$$m\lambda^2 Xe^{\lambda t} + c\lambda Xe^{\lambda t} + kXe^{\lambda t} = 0 \ , \tag{23}$$

$$(m\lambda^2 + c\lambda + k)Xe^{\lambda t} = 0 \ , \tag{24}$$

Note that m, c, k, λ and X do *not* depend on time. For equation (24) to be true for all time,

$$(m\lambda^2 + c\lambda + k)X = 0 \ . \tag{25}$$

Equation (25) is trivially satisfied if $X = 0$. The *non-trivial solution* is $m\lambda^2 + c\lambda + k = 0$. This is a quadratic equation in λ which has the roots,

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}. \tag{26}$$

The solution to a homogeneous second order ordinary differential equation requires two independent initial conditions: an initial displacement and an initial velocity. These two independent initial conditions are used to determine the coefficients, X and X^* (or A and B, or a and b) of the two linearly independent solutions corresponding to λ_1 and λ_2.

The amount of damping, c, qualitatively affects the quadratic roots, $\lambda_{1,2}$, and the free response solutions.

- **Case 1** $c = 0$ "undamped"

  If the system has no damping, $c = 0$, and

  $$\lambda_{1,2} = \pm i\sqrt{k/m} = \pm i\omega_{\rm n} \ . \tag{27}$$

  This is called the *natural frequency* of the system. Undamped systems oscillate freely at their natural frequency, $\omega_{\rm n}$. The solution in this case is

  $$x(t) = Xe^{i\omega_{\rm n}t} + X^*e^{-i\omega_{\rm n}t} = a\cos\omega_{\rm n}t + b\sin\omega_{\rm n}t \ , \tag{28}$$

which is a *real-valued* function. The amplitudes depend on the initial displacement, d_o, and the initial velocity, v_o.

- **Case 2** $c = c_c$ "critically damped"
  If $(c/(2m))^2 = k/m$, or, equivalently, if $c = 2\sqrt{mk}$, then the discriminant of equation (26) is zero, This special value of damping is called the *critical damping* rate, c_c,
  $$c_c = 2\sqrt{mk} \ . \tag{29}$$
  The ratio of the actual damping rate to the critical damping rate is called the *damping ratio*, ζ.
  $$\zeta = \frac{c}{c_c} \ . \tag{30}$$
  The two roots of the quadratic equation are real and are repeated at
  $$\lambda_1 = \lambda_2 = -c/(2m) = -c_c/(2m) = -2\sqrt{mk}/(2m) = -\omega_n \ , \tag{31}$$
  and the two basic solutions are equal to each other, $e^{\lambda_1 t} = e^{\lambda_2 t}$. In order to admit solutions for arbitrary initial displacements and velocities, the solution in this case is
  $$x(t) = x_1 \ e^{-\omega_n t} + x_2 \ t \ e^{-\omega_n t} \ . \tag{32}$$
  where the real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o. Details regarding this special case are in the next section.

- **Case 3** $c > c_c$ "over-damped"
  If the damping is greater than the critical damping, then the roots, λ_1 and λ_2 are distinct and real. If the system is over-damped it will not oscillate freely. The solution is
  $$x(t) = x_1 \ e^{\lambda_1 t} + x_2 \ e^{\lambda_2 t} \ , \tag{33}$$
  which can also be expressed using hyperbolic sine and hyperbolic cosine functions. The real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o.

- **Case 4** $0 < c < c_{\rm c}$ "under-damped"
  If the damping rate is positive, but less than the critical damping rate, the system will oscillate freely from some initial displacement and velocity. The roots are complex conjugates, $\lambda_1 = \lambda_2^*$, and the solution is

$$x(t) = X \; e^{\lambda t} + X^* \; e^{\lambda^* t} \; , \tag{34}$$

  where the complex amplitude depends on the initial displacement, d_o, and the initial velocity, v_o.

We can re-write the dynamic equations of motion using the new dynamic variables for natural frequency, $\omega_{\rm n}$, and damping ratio, ζ. Note that

$$\frac{c}{m} = c\frac{\sqrt{k}}{\sqrt{k}}\frac{1}{\sqrt{m}\sqrt{m}} = \frac{c}{\sqrt{k}\sqrt{m}}\frac{\sqrt{k}}{\sqrt{m}} = 2\frac{c}{2\sqrt{km}}\sqrt{\frac{k}{m}} = 2\zeta\omega_{\rm n}. \tag{35}$$

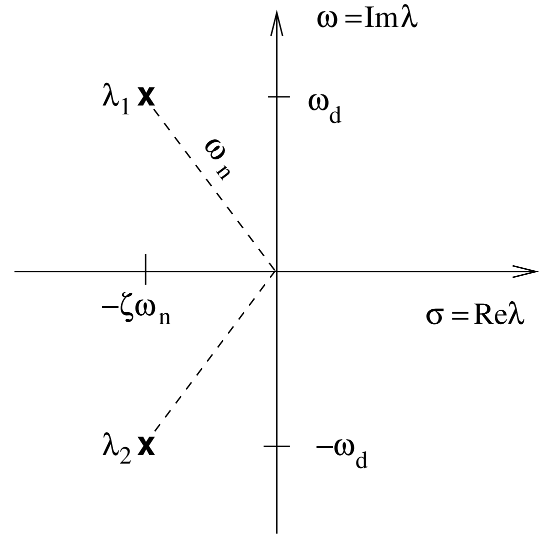
$$\begin{aligned} m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= f(t), & (36)\\ \ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) &= \frac{1}{m}f(t), & (37)\\ \ddot{x}(t) + 2\zeta\omega_{\rm n}\;\dot{x}(t) + \omega_{\rm n}^2\;x(t) &= \frac{1}{m}f(t), & (38)\end{aligned}$$

The expression for the roots $\lambda_{1,2}$, can also be written in terms of $\omega_{\rm n}$ and ζ.

$$\begin{aligned} \lambda_{1,2} &= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \; , & (39)\\ &= -\zeta\omega_{\rm n} \pm \sqrt{(\zeta\omega_{\rm n})^2 - \omega_{\rm n}^2} \; , & (40)\\ &= -\zeta\omega_{\rm n} \pm \omega_{\rm n}\sqrt{\zeta^2 - 1} \; . & (41)\end{aligned}$$

Some useful facts about the roots λ_1 and λ_2 are:

- $\lambda_1 + \lambda_2 = -2\zeta\omega_{\rm n}$
- $\lambda_1 - \lambda_2 = 2\ \omega_{\rm n}\ \sqrt{\zeta^2 - 1}$
- $\omega_{\rm n}^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 - \frac{1}{4}(\lambda_1 - \lambda_2)^2$
- $\omega_{\rm n} = \sqrt{\lambda_1 \lambda_2}$
- $\zeta = -(\lambda_1 + \lambda_2)/(2\omega_{\rm n})$



2.1 Free response of critically-damped systems

The solution to a homogeneous second order ordinary differential equation requires two independent initial conditions: an initial displacement and an initial velocity. These two initial conditions are used to determine the coefficients of the two linearly independent solutions corresponding to λ_1 and λ_2. If $\lambda_1 = \lambda_2$, then the solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are not independent. In fact, they are identical. In such a case, a new trial solution can be determined as follows. Assume the second solution has the form

$$x(t) = u(t) x_2 e^{\lambda_2 t} \ , \qquad (42)$$

$$\dot{x}(t) = \dot{u}(t) x_2 e^{\lambda_2 t} + u(t) \lambda_2 x_2 e^{\lambda_2 t} \ , \qquad (43)$$

$$\ddot{x}(t) = \ddot{u}(t) x_2 e^{\lambda_2 t} + 2\dot{u}(t)\lambda_2 x_2 e^{\lambda_2 t} + u(t)\lambda_2^2 x_2 e^{\lambda_2 t} \qquad (44)$$

substitute these expressions into

$$\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + \omega_{\rm n}^2 x(t) = 0 \ ,$$

collect terms, and divide by $x_2 e^{\lambda_2 t}$, to get

$$\ddot{u}(t) + 2\omega_{\rm n}(\zeta - 1)\dot{u}(t) + 2\omega_{\rm n}^2(1 - \zeta)u(t) = 0$$

or $\ddot{u}(t) = 0$ (since $\zeta = 1$). If the acceleration of $u(t)$ is zero then the velocity of $u(t)$ must be constant, $\dot{u}(t) = C$, and $u(t) = Ct$, from which the new trial solution is found.

$$x(t) = u(t) x_2 e^{\lambda_2 t} = x_2\ t\ e^{\lambda_2 t} \ .$$

So, using the complete trial solution $x(t) = x_1 e^{\lambda t} + x_2 t e^{\lambda t}$, and incorporating initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$, the free response of a critically-damped system is:

$$x(t) = d_o \; e^{-\omega_{\rm n} t} + (v_o + \omega_{\rm n} d_o) \; t \; e^{-\omega_{\rm n} t} \; . \tag{45}$$

2.2 Free response of underdamped systems

If the system is under-damped, then $\zeta < 1$, $\sqrt{\zeta^2 - 1}$ is imaginary, and

$$\lambda_{1,2} = -\zeta\omega_{\rm n} \pm i\omega_{\rm n}\sqrt{|\zeta^2 - 1|} = \sigma \pm i\omega. \tag{46}$$

The frequency $\omega_{\rm n}\sqrt{|\zeta^2 - 1|}$ is called the *damped natural frequency*, $\omega_{\rm d}$,

$$\omega_{\rm d} = \omega_{\rm n}\sqrt{|\zeta^2 - 1|} \; . \tag{47}$$

It is the frequency at which under-damped SDOF systems oscillate freely, With these new dynamic variables (ζ, $\omega_{\rm n}$, and $\omega_{\rm d}$) we can re-write the solution to the damped free response,

$$x(t) = e^{-\zeta\omega_{\rm n} t}(a \cos \omega_{\rm d} t + b \sin \omega_{\rm d} t), \tag{48}$$

$$= X e^{\lambda t} + X^* e^{\lambda^* t}. \tag{49}$$

Now we can solve for X, (or, equivalently, A and B) in terms of the initial conditions. At the initial point in time, $t = 0$, the position of the mass is $x(0) = d_o$ and the velocity of the mass is $\dot{x}(0) = v_o$.

$$x(0) = d_o = X e^{\lambda \cdot 0} + X^* e^{\lambda^* \cdot 0} \tag{50}$$

$$= X + X^* \tag{51}$$

$$= (A + iB) + (A - iB) = 2A = a. \tag{52}$$

$$\dot{x}(0) = v_o = \lambda X e^{\lambda \cdot 0} + \lambda^* X^* e^{\lambda^* \cdot 0}, \tag{53}$$

$$= \lambda X + \lambda^* X^*, \tag{54}$$

$$= (\sigma + i\omega_{\rm d})(A + iB) + (\sigma - i\omega_{\rm d})(A - iB), \tag{55}$$

$$= \sigma A + i\omega_{\rm d} A + i\sigma B - \omega_{\rm d} B + \sigma A - i\omega_{\rm d} A - i\sigma B - \omega_{\rm d} B, \tag{56}$$

$$= 2\sigma A - 2\omega B \tag{57}$$

$$= -\zeta\omega_{\rm n} \; d_o - 2 \; \omega_{\rm d} \; B, \tag{58}$$

from which we can solve for B and b,

$$B = -\frac{v_o + \zeta\omega_{\rm n} d_o}{2\omega_{\rm d}} \qquad \text{and} \qquad b = \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}} \ . \tag{59}$$

Putting this all together, the free response of an underdamped system to an arbitrary initial condition, $x(0) = d_o, \dot{x}(0) = v_o$ is

$$x(t) = e^{-\zeta\omega_{\rm n} t} \left(d_o \cos\omega_{\rm d} t + \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}} \sin\omega_{\rm d} t \right) . \tag{60}$$

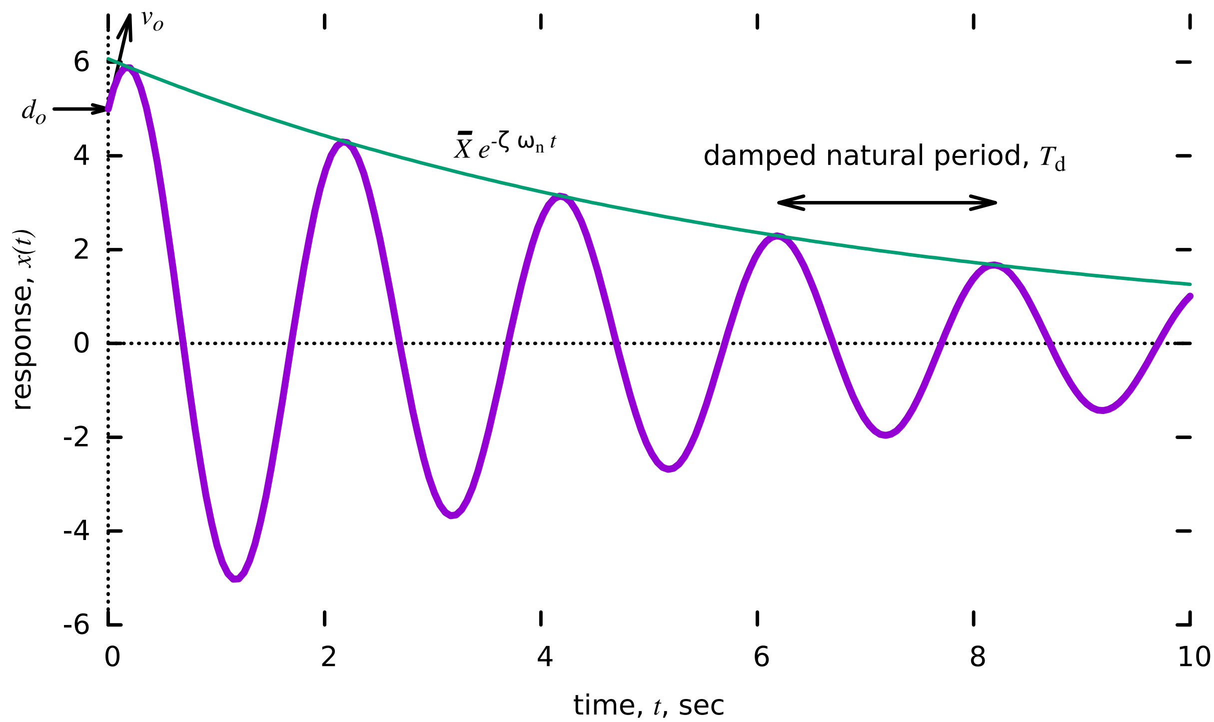


Figure 5. Free response of an under-damped oscillator to an initial displacement and velocity.

2.3 Free response of over-damped systems

If the system is over-damped, then $\zeta > 1$, and $\sqrt{\zeta^2 - 1}$ is real, and the roots are both real and negative

$$\lambda_{1,2} = -\zeta\omega_{\rm n} \pm \omega_{\rm n}\sqrt{\zeta^2 - 1} = \sigma \pm \omega_{\rm d}. \tag{61}$$

Substituting the initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$ into the solution (equation (33)), and solving for the coefficients results in

$$x_1 = \frac{v_o + d_o(\zeta\omega_{\rm n} + \omega_{\rm d})}{2\omega_{\rm d}} , \tag{62}$$

$$x_2 = d_o - x_1 . \tag{63}$$

Substituting the hyperbolic sine and hyperbolic cosine expressions for the exponentials results in

$$x(t) = e^{-\zeta\omega_{\rm n}t}\left(d_o \ \cosh\omega_{\rm d}t + \frac{v_o + \zeta\omega_{\rm n}d_o}{\omega_{\rm d}} \ \sinh\omega_{\rm d}t\right) . \tag{64}$$

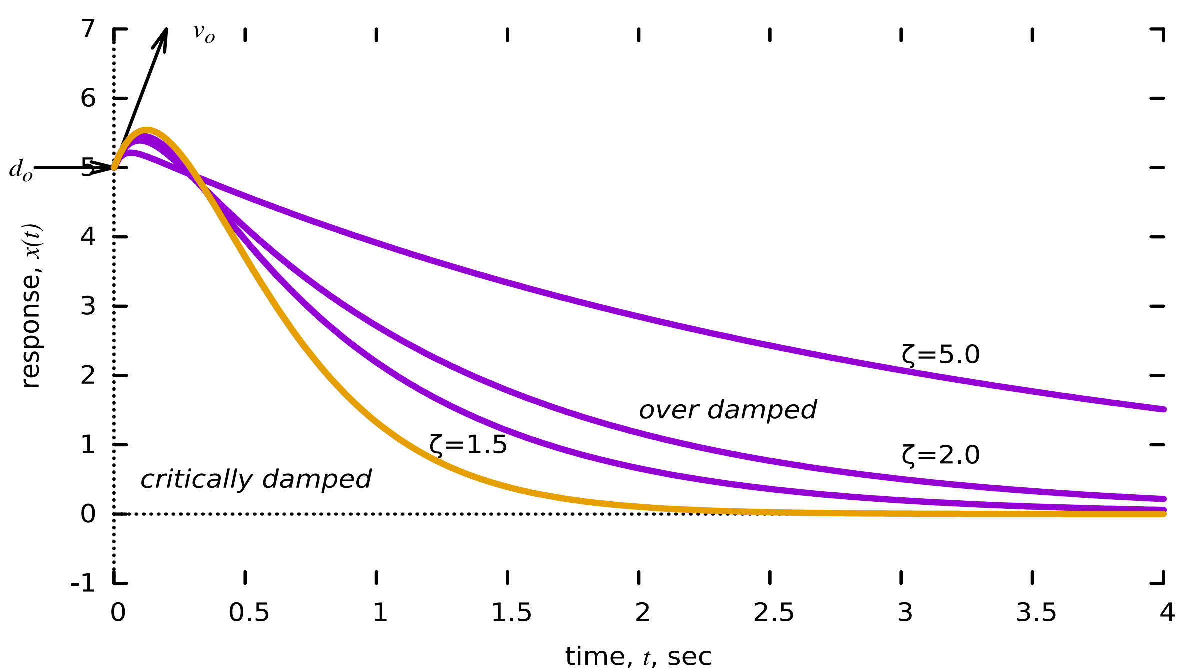


Figure 6. Free response of critically-damped (yellow) and over-damped (violet) oscillators to an initial displacement and velocity.

The undamped free response can be found as a special case of the under-damped free response. While special solutions exist for the critically damped response, this response can also be found as limiting cases of the under-damped or over-damped responses.

2.4 Finding the natural frequency from self-weight displacement

Consider a spring-mass system in which the mass is loaded by gravity, g. The static displacement D_{st} is related to the natural frequency by the constant of gravitational acceleration.

$$D_{\mathrm{st}} = mg/k = g/\omega_{\mathrm{n}}^2 \tag{65}$$

2.5 Finding the damping ratio from free response

Consider the value of two peaks of the free response of an under-damped system, separated by n cycles of motion

$$x_1 = x(t_1) \;=\; e^{-\zeta\omega_{\mathrm{n}}t_1}(A) \tag{66}$$

$$x_{1+n} = x(t_{1+n}) \;=\; e^{-\zeta\omega_{\mathrm{n}}t_{1+n}}(A) = e^{-\zeta\omega_{\mathrm{n}}(t_1+2n\pi/\omega_{\mathrm{d}})}(A) \tag{67}$$

The ratio of these amplitudes is

$$\frac{x_1}{x_{1+n}} = \frac{e^{-\zeta\omega_{\mathrm{n}}t_1}}{e^{-\zeta\omega_{\mathrm{n}}(t_1+2n\pi/\omega_{\mathrm{d}})}} = \frac{e^{-\zeta\omega_{\mathrm{n}}t_1}}{e^{-\zeta\omega_{\mathrm{n}}t_1}e^{-2n\pi\zeta\omega_{\mathrm{n}}/\omega_{\mathrm{d}}}} = e^{2n\pi\zeta/\sqrt{1-\zeta^2}}, \tag{68}$$

which is independent of ω_{n} and ω_{d}. Defining the *log decrement* $\delta(\zeta)$ as $\ln(x_1/x_{1+n})/n$,

$$\delta(\zeta) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{69}$$

and, inversely,

$$\zeta(\delta) = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \approx \frac{\delta}{2\pi} \tag{70}$$

where the approximation is accurate to within 3% for $\zeta < 0.2$ and is accurate to within 1.5% for $\zeta < 0.1$.

2.6 Summary

To review, some of the important expressions relating to the free response of a single degree of freedom oscillator are:

$$\bar{X}\cos(\omega t + \theta) = a\cos(\omega t) + b\sin(\omega t) = Xe^{+i\omega t} + X^* e^{-i\omega t}$$

$$\bar{X} = \sqrt{a^2 + b^2}; \quad \tan(\theta) = -b/a; \quad X = A + iB; \quad A = a/2; B = -b/2;$$

$$\left.\begin{array}{r} m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \\ \ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + \omega_{\rm n}^2 x(t) = 0 \end{array}\right\} \qquad x(0) = d_o, \ \dot{x}(0) = v_o$$

$$\omega_{\rm n} = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{c_{\rm c}} = \frac{c}{2\sqrt{mk}} \qquad \omega_{\rm d} = \omega_{\rm n}\sqrt{|\zeta^2 - 1|}$$

$$x(t) = e^{-\zeta\omega_{\rm n} t}\left(d_o \cos\omega_{\rm d} t + \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}} \sin\omega_{\rm d} t\right) \qquad (0 \le \zeta < 1)$$

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{1+n}}\right) \qquad \zeta(\delta) = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx \frac{\delta}{2\pi}$$

3 Response of systems with mass, stiffness, and damping to sinusoidal forcing

When subject to simple harmonic forcing with a forcing frequency ω, dynamic systems initially respond with a combination of a transient response at a frequency $\omega_{\rm d}$ and a steady-state response at a frequency ω. The transient response at frequency $\omega_{\rm d}$ decays with time, leaving the steady state response at a frequency equal to the forcing frequency, ω. This section examines three ways of applying forcing: forcing applied directly to the mass, inertial forcing applied through motion of the base, and forcing from a rotating eccentric mass.

3.1 Direct Force Excitation

If the SDOF system is dynamically forced with a sinusoidal forcing function, then $f(t) = \bar{F}\cos(\omega t)$, where ω is the frequency of the forcing, in radians per second. If $f(t)$ is persistent, then after several cycles the system will respond only at the frequency of the external forcing, ω. Let's suppose that this *steady-state response* is described by the function

$$x(t) = a\cos\omega t + b\sin\omega t, \tag{71}$$

then

$$\dot{x}(t) = \omega(-a\sin\omega t + b\cos\omega t), \tag{72}$$

and

$$\ddot{x}(t) = \omega^2(-a\cos\omega t - b\sin\omega t). \tag{73}$$

Substituting this trial solution into equation (6), we obtain

$$\begin{aligned} m\omega^2 &\quad (-a\cos\omega t - b\sin\omega t) + \\ c\omega &\quad (-a\sin\omega t + b\cos\omega t) + \\ k &\quad (a\cos\omega t + b\sin\omega t) = \bar{F}\cos\omega t. \end{aligned} \tag{74}$$

Equating the sine terms and the cosine terms

$$(-m\omega^2 a + c\omega b + ka)\cos\omega t \;=\; \bar{F}\cos\omega t \tag{75}$$

$$(-m\omega^2 b - c\omega a + kb)\sin\omega t \;=\; 0, \tag{76}$$

which is a set of two equations for the two unknown constants, a and b,

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{F} \\ 0 \end{bmatrix}, \tag{77}$$

for which the solution is

$$a(\omega) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \bar{F} \tag{78}$$

$$b(\omega) = \frac{c\omega}{(k - m\omega^2)^2 + (c\omega)^2} \bar{F}. \tag{79}$$

Referring to equations (7) and (12) in section 1.1, the forced vibration solution (equation (71)) may be written

$$x(t) = a(\omega)\cos\omega t + b(\omega)\sin\omega t = \bar{X}(\omega)\cos\left(\omega t + \theta(\omega)\right). \tag{80}$$

The angle θ is the phase between the force $f(t)$ and the response $x(t)$, and

$$\tan(\theta(\omega)) = -\frac{b(\omega)}{a(\omega)} = -\frac{c\omega}{k - m\omega^2} \tag{81}$$

Note that $-\pi < \theta(\omega) < 0$ for all positive values of ω, meaning that the displacement response, $x(t)$, always lags the external forcing, $\bar{F}\cos(\omega t)$. The ratio of the response amplitude $\bar{X}(\omega)$ to the forcing amplitude $\bar{F}$ is

$$\frac{\bar{X}(\omega)}{\bar{F}} = \frac{\sqrt{a^2(\omega) + b^2(\omega)}}{\bar{F}} = \frac{1}{\sqrt{(k - m\omega^2) + (c\omega)^2}}. \tag{82}$$

This equation shows how the response amplitude $\bar{X}$ depends on the amplitude of the forcing $\bar{F}$ and the frequency of the forcing ω, and has units of flexibility.

Let's re-derive this expression using complex exponential notation! The equations of motion are

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \bar{F}\cos\omega t = F(\omega)e^{i\omega t} + F^*(\omega)e^{-i\omega t}. \tag{83}$$

In a solution of the form, $x(t) = X(\omega)e^{i\omega t} + X^*(\omega)e^{-i\omega t}$, the coefficient $X(\omega)$ corresponds to the positive exponents (positive frequencies), and $X^*(\omega)$ corresponds to negative exponents (negative frequencies). Positive exponent coefficients and negative exponent coefficients are independent and may be found separately. Considering the positive exponent solution, the forcing is expressed as $F(\omega)e^{i\omega t}$ and the partial solution $X(\omega)e^{i\omega t}$ is substituted into the forced equations of motion, resulting in

$$(-m\omega^2 + ci\omega + k)\ X(\omega)\ e^{i\omega t} = F(\omega)\ e^{i\omega t}\ , \tag{84}$$

from which

$$\frac{X(\omega)}{F(\omega)} = \frac{1}{(k - m\omega^2) + i(c\omega)}\ , \tag{85}$$

which is complex-valued. This complex function has a magnitude

$$\left|\frac{X(\omega)}{F(\omega)}\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\ , \tag{86}$$

the same as equation (82) but derived using $e^{i\omega t}$ in just three simple lines.

Equation (85) may be written in terms of the dynamic variables, $\omega_{\rm n}$ and ζ. Dividing the numerator and the denominator of equation (82) by k, and noting that F/k is a static displacement, $x_{\rm st}$, we obtain

$$\frac{X(\omega)}{F(\omega)} = \frac{1/k}{\left(1 - \frac{m}{k}\omega^2\right) + i\left(\frac{c}{k}\omega\right)}, \tag{87}$$

$$X(\omega) = \frac{F(\omega)/k}{\left(1 - \left(\frac{\omega}{\omega_{\rm n}}\right)^2\right) + i\left(2\zeta\frac{\omega}{\omega_{\rm n}}\right)}, \tag{88}$$

$$\frac{X(\Omega)}{x_{\rm st}} = \frac{1}{(1 - \Omega^2) + i\,(2\zeta\Omega)}, \tag{89}$$

$$\frac{\bar{X}(\Omega)}{x_{\rm st}} = \frac{1}{\sqrt{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}}, \tag{90}$$

where the frequency ratio Ω is the ratio of the forcing frequency to the natural frequency, $\Omega = \omega/\omega_{\rm n}$. This equation is called the *dynamic amplification factor.* It is the factor by which displacement responses are amplified due to the fact that the external forcing is dynamic, not static. See Figure 7.

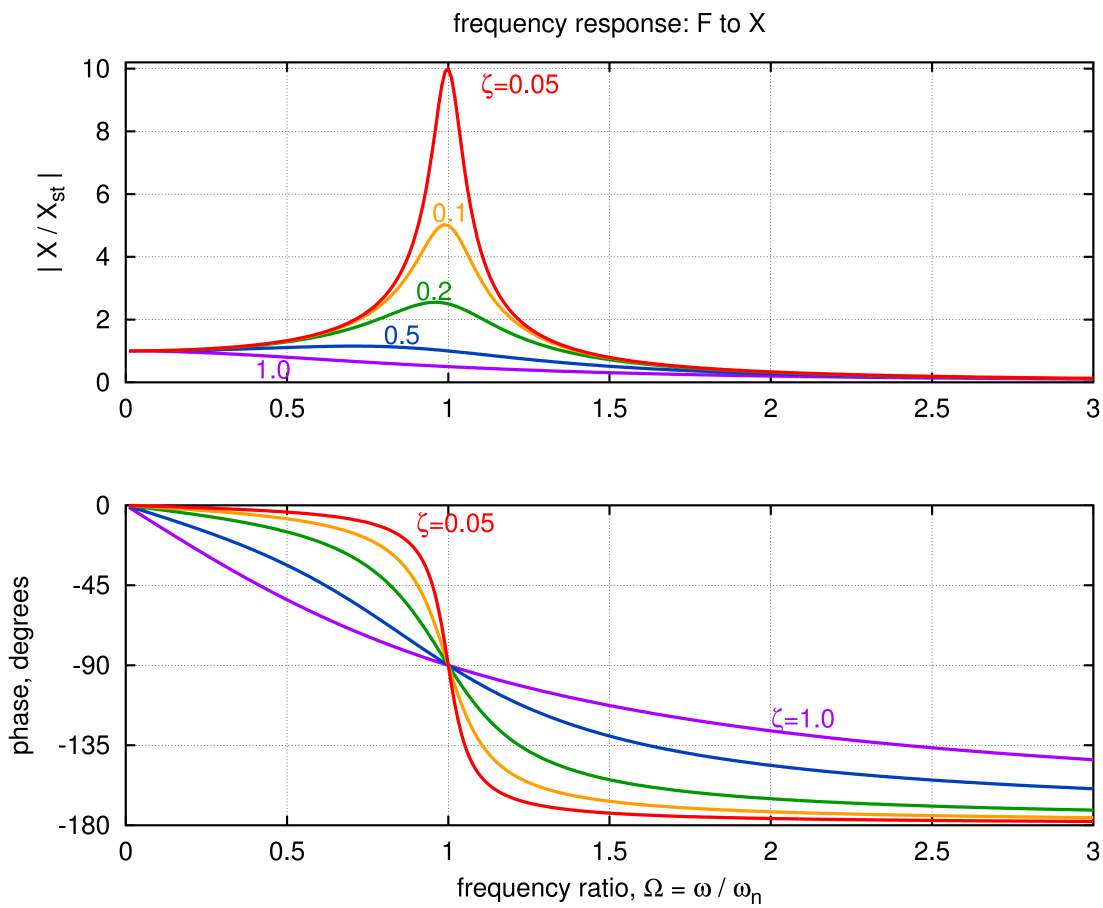


Figure 7. The dynamic amplification factor for external forcing, $\bar{X}/x_{\rm st}$, equation (89).

To summarize, the steady state response of a simple oscillator directly excited by a harmonic force, $f(t) = \bar{F}\cos\omega t$, may be expressed in the form of equation (7)

$$x(t) = \frac{\bar{F}/k}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} \ \cos(\omega t + \theta) \ , \quad \tan\theta = \frac{-2\zeta\Omega}{1-\Omega^2} \tag{91}$$

or, equivalently, in the form of equation (8)

$$x(t) = \frac{\bar{F}/k}{(1-\Omega^2)^2 + (2\zeta\Omega)^2} \ [\ (1-\Omega^2)\cos\omega t + (2\zeta\Omega)\sin\omega t \ ] \ , \tag{92}$$

where $\Omega = \omega/\omega_{\rm n}$.

3.2 Support Acceleration Excitation

When the dynamic loads are caused by motion of the supports (or the ground, as in an earthquake) the forcing on the structure is the inertial force resisting the ground acceleration, which equals the mass of the structure times the ground acceleration, $f(t) = -m\ddot{z}(t)$.

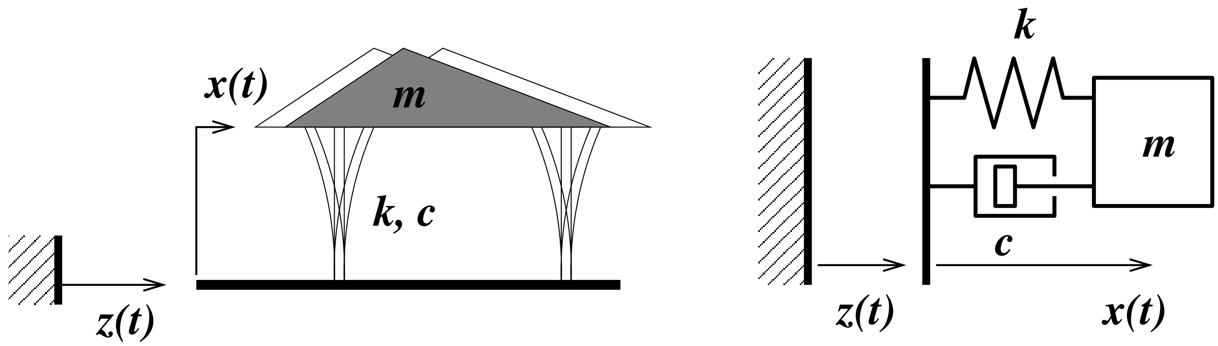


Figure 8. The proto-typical SDOF oscillator subjected to base motions, $z(t)$

$$
\begin{aligned}
m\,(\ \ddot{x}(t) + \ddot{z}(t)\ ) + c\dot{x}(t) + kx(t) &= 0 && (93)\\
m\ddot{x}(t) + c\dot{x}(t) + kx(t) &= -m\ddot{z}(t) && (94)\\
\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + \omega_{\rm n}^2 x(t) &= -\ddot{z}(t) && (95)
\end{aligned}
$$

Note that equation (95) is independent of mass. Systems of different masses but with the same natural frequency and damping ratio have the same behavior and respond in exactly the same way to the same support motion.

If the ground displacements are sinusoidal $z(t) = \bar{Z}\cos\omega t$, then the ground accelerations are $\ddot{z}(t) = -\bar{Z}\omega^2\cos\omega t$, and $f(t) = m\bar{Z}\omega^2\cos\omega t$. Using the complex exponential formulation, we can find the steady state response as a function of the frequency of the ground motion, ω.

$$
m\ddot{x}(t) + c\dot{x}(t) + kx(t) = m\bar{Z}\omega^2\cos\omega t = mZ(\omega)\omega^2 e^{i\omega t} + mZ^*(\omega)\omega^2 e^{-i\omega t} \quad (96)
$$

The steady-state response can be expressed as the sum of independent complex exponentials, $x(t) = X(\omega)e^{i\omega t} + X^*(\omega)e^{-i\omega t}$. The positive exponent parts are independent of the negative exponent parts and can be analyzed separately.

Assuming persistent excitation and ignoring the transient resposne (the particular part of the solution), the response will be harmonic. Considering the "positive exponent" part of the forcing $mZ(\omega)\omega^2 e^{i\omega t}$, the "positive exponent" part of the steady-state response will have a form $Xe^{i\omega t}$. Substituting these expressions into the differential equation (96), collecting terms, and factoring out the exponential $e^{i\omega t}$, the frequency response function is

$$\begin{aligned} \frac{X(\omega)}{Z(\omega)} &= \frac{m\omega^2}{(k - m\omega^2) + i(c\omega)} , \\ &= \frac{\Omega^2}{(1-\Omega^2) + i(2\zeta\Omega)} \end{aligned} \tag{97}$$

where $\Omega = \omega/\omega_{\rm n}$ (the forcing frequency over the natural frequency), and

$$\left| \frac{X(\Omega)}{Z(\Omega)} \right| = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} \tag{98}$$

See Figure 9.

Finally, let's consider the motion of the mass with respect to a fixed point. This is called the total motion and is the sum of the base motion plus the motion relative to the base, $z(t) + x(t)$.

$$\begin{aligned} \frac{X+Z}{Z} = \frac{X}{Z} + 1 &= \frac{(1-\Omega^2) + i(2\zeta\Omega) + \Omega^2}{(1-\Omega^2) + i(2\zeta\Omega)} \\ &= \frac{1 + i(2\zeta\Omega)}{(1-\Omega^2) + i(2\zeta\Omega)} \end{aligned} \tag{99}$$

and

$$\left| \frac{X+Z}{Z} \right| = \frac{\sqrt{1 + (2\zeta\Omega)^2}}{\sqrt{(1-\Omega^2)^2 + (2\zeta\Omega)^2}} = \mathrm{Tr}(\Omega,\zeta). \tag{100}$$

This function is called the *transmissibility ratio*, $\mathrm{Tr}(\Omega,\zeta)$. It determines the ratio between the total response amplitude $\overline{X+Z}$ and the base motion $\bar{Z}$. See figure 10.

For systems that have a longer natural period (lower natural frequency) than the period (frequency) of the support motion, (i.e., $\Omega > \sqrt{2}$), the transmissibility ratio is less than "1" especialy for low values of damping ζ. In such systems the motion of the mass is *less than* the motion of the supports and we say that the mass is *isolated* from motion of the supports.

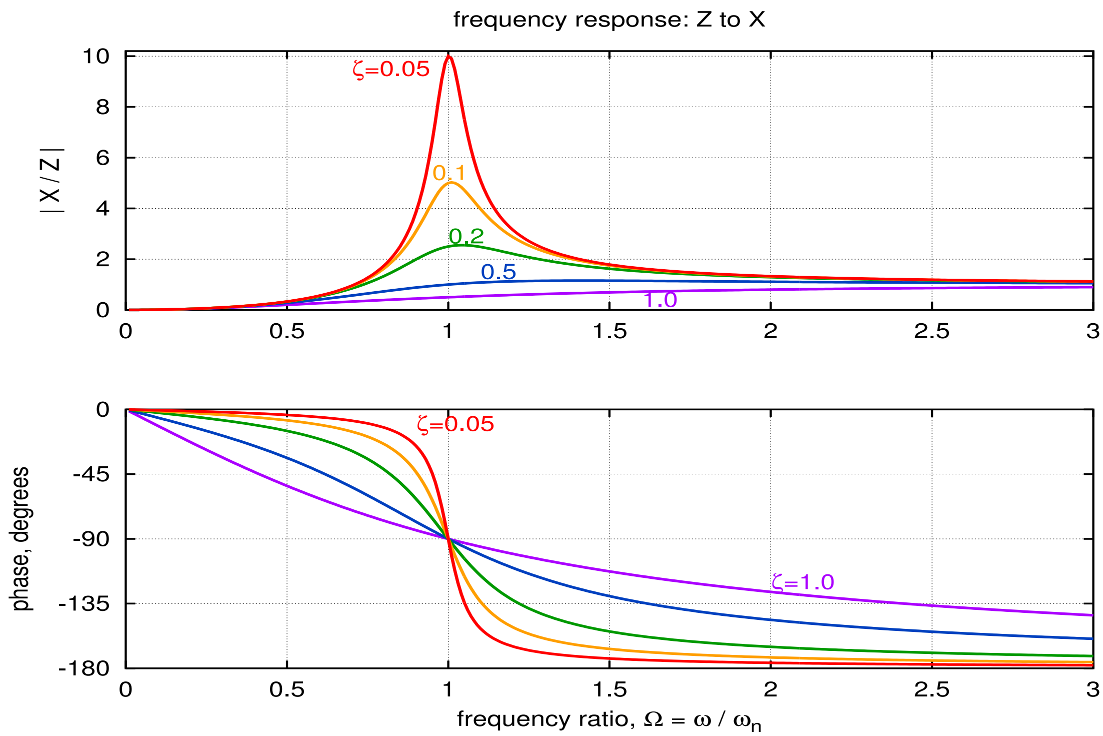


Figure 9. The dynamic amplification factor for base-excitation, $\bar{X}/\bar{Z}$, equation (97).

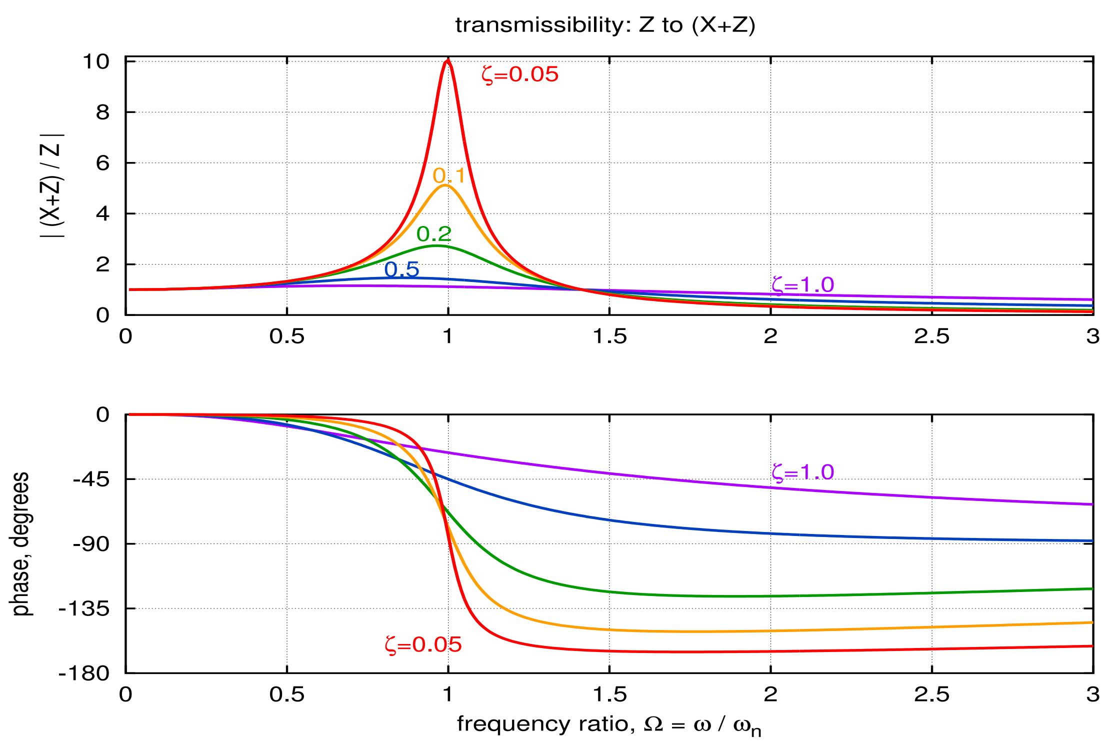


Figure 10. The transmissibility ratio $|(X+Z)/Z| = \mathrm{Tr}(\Omega, \zeta)$, equation (99).

3.3 Eccentric-Mass Excitation

Another type of sinusoidal forcing which is important to machine vibration arises from the rotation of an eccentric mass. Consider the system shown in Figure 11 in which a mass μm is attached to the primary mass m via a rigid link of length r and rotates at an angular velocity ω. In this single degree of freedom analysis, the motion of the primary mass is constrained to lie along the x coordinate and the forcing of interest is the x-component of the reactive centrifugal force. This component is $\mu m r \omega^2 \cos(\omega t)$ where the angle ωt is the counter-clockwise angle from the x coordinate. The equation of motion with

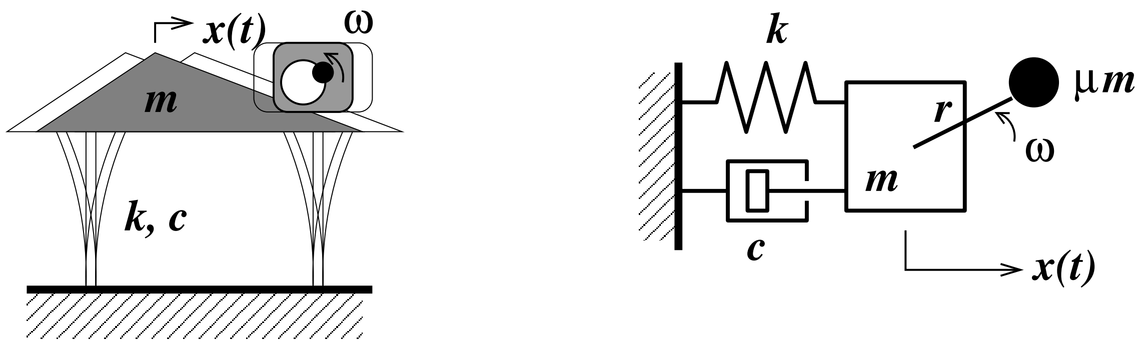


Figure 11. The proto-typical SDOF oscillator subjected to eccentric-mass excitation.

this forcing is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \mu m r \omega^2 \ \cos(\omega t) \tag{101}$$

This expression is simply analogous to equation (83) in which $\bar{F} = \mu \ m \ r \ \omega^2$. With a few substitutions, the frequency response function is found to be

$$\frac{X}{r} = \frac{\mu \ \Omega^2}{(1 - \Omega^2) + i\,(2\zeta\Omega)}, \tag{102}$$

which is completely analogous to equation (97). The plot of the frequency response function of equation (102) is simply proportional to the function plotted in Figure 9. The magnitude of the dynamic force transmitted between a machine supported on dampened springs and the base, $|f_{\rm T}|$, is related to the transmissibility ratio.

$$\frac{|f_{\rm T}|}{kr} = \mu\Omega^2 \ {\rm Tr}(\Omega, \zeta) \tag{103}$$

Unlike the transmissibility ratio asymptotically approaches "0" with increasing Ω, the vibratory force transmitted from eccentric mass excitation is "0" when $\Omega = 0$ but increase with Ω for $\Omega > \sqrt{2}$. This increasing effect is significant for $\zeta > 0.2$, as shown in Figure 12.

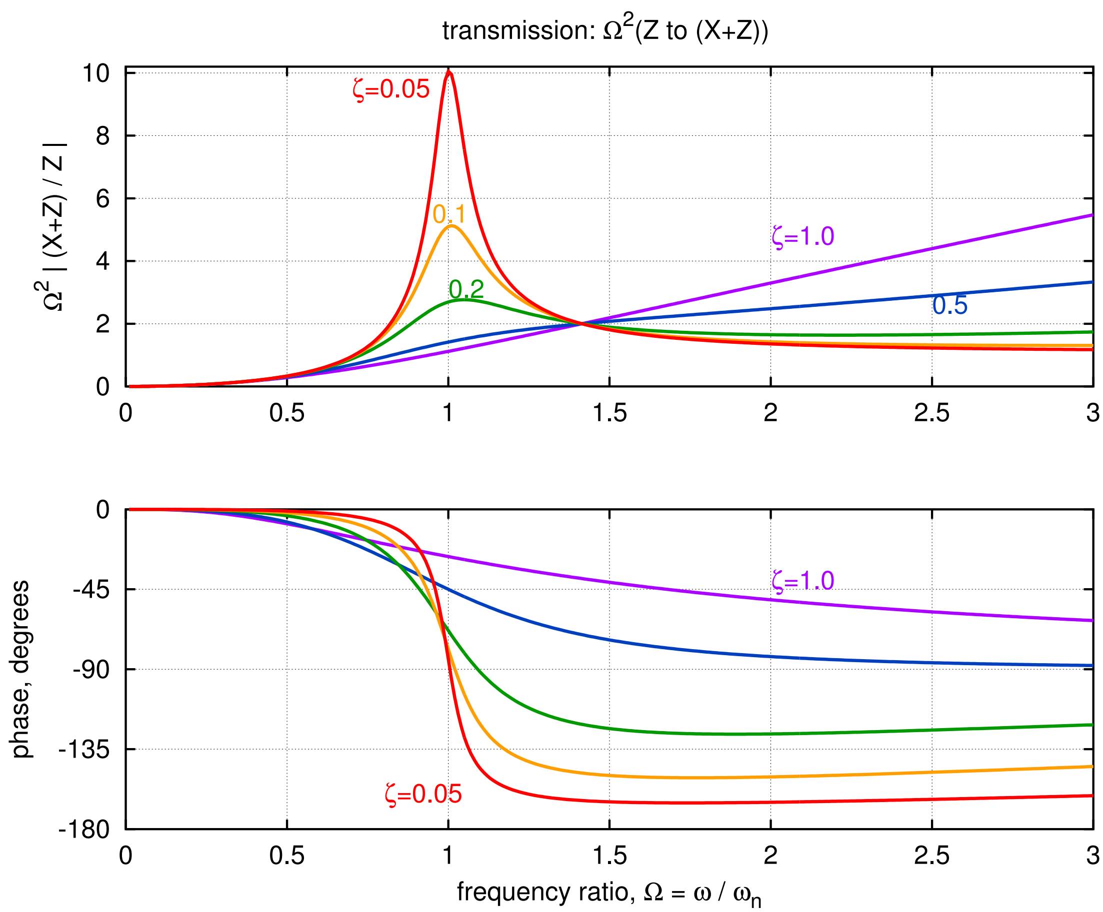


Figure 12. The transmission ratio $\Omega^2 \mathrm{Tr}(\Omega, \zeta)$, equation (103).

3.4 Finding the damping from the peak of the frequency response function

For lightly damped systems, the frequency ratio of the resonant peak, the amplification of the resonant peak, and the width of the resonant peak are functions to of the damping ratio only. Consider two frequency ratios Ω_1 and Ω_2 such that $|H(\Omega_1,\zeta)|^2 = |H(\Omega_2,\zeta)|^2 = |H|^2_{\mathrm{peak}}/2$ where $|H(\Omega,\zeta)|$ is one of the frequency response functions described in earlier sections. The frequency ratio corresponding to the peak of these functions Ω_{peak}, and the value of the peak of these functions, $|H|^2_{\mathrm{peak}}$ are given in Table 1. Note that the peak coordinate depends only upon the damping ratio, ζ.

Since $\Omega_2^2 - \Omega_1^2 = (\Omega_2 - \Omega_1)(\Omega_2 + \Omega_1)$ and since $\Omega_2 + \Omega_1 \approx 2$,

$$\zeta \approx \frac{\Omega_2 - \Omega_1}{2} \tag{104}$$

which is called the "half-power bandwidth" formula for damping. For the first, second, and fourth frequency response functions listed in Table 1 the approximation is accurate to within 5% for $\zeta < 0.20$ and is accurate to within 1% for $\zeta < 0.10$.

Table 1. Peak coordinates for various frequency response functions.

$H(\Omega,\zeta)$	Ω_{peak}	$\lvert H\rvert^2_{\mathrm{peak}}$	$\Omega_2^2 - \Omega_1^2$
$\frac{1}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\sqrt{1-2\zeta^2}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$	$4\zeta\sqrt{1-\zeta^2}$
$\frac{i\Omega}{(1-\Omega^2)+i(2\zeta\Omega)}$	1	$\frac{1}{4\zeta^2}$	$4\zeta\sqrt{1+\zeta^2}$
$\frac{\Omega^2}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\frac{1}{\sqrt{1-2\zeta^2}}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$	$\frac{4\zeta\sqrt{1-\zeta^2}}{1-8\zeta^2(1-\zeta^2)}$
$\frac{1+i(2\zeta\Omega)}{(1-\Omega^2)+i(2\zeta\Omega)}$	$\frac{\left((1+8\zeta^2)^{1/2}-1\right)^{1/2}}{2\zeta}$	$\frac{8\zeta^4}{8\zeta^4-4\zeta^2-1+\sqrt{1+8\zeta^2}}$	ouch.