

Vibrations of Two Degree of Freedom Systems

Introduction

A two degree of freedom system is one that requires two coordinates to completely describe its equation of motion. These coordinates are called generalized coordinates when they are independent of each other. Thus system with two degrees of freedom will have two equation of motion and hence has two frequencies.

A two degree freedom system differs from a single degree of freedom system in that it has two natural frequencies and for each of these natural frequencies there correspond a natural state of vibration with a displacement configuration known as **NORMAL MODE**. Mathematical terms related to these quantities are known as Eigen values and Eigen vectors. These are established from the two simultaneous equation of motion of the system and posses certain dynamic properties associated.

A system having two degrees of freedom are important in as far as they introduce to the coupling phenomenon where the motion of any of the two independent coordinates depends also on the motion of the other coordinate through the coupling spring and damper. The free vibration of two degrees of freedom system at any point is a combination of two harmonics of these two natural frequencies.

Under certain condition, during free vibrations any point in a system may execute harmonic vibration at any of the two natural frequencies and the amplitude are related in a specific manner and the configuration is known as **NORMALMODE or PRINCIPAL MODE** of vibration. Thus system with two degrees of freedom has two normal modes of vibration corresponding two natural frequencies.

Free vibrations of two degrees of freedom system:

Consider an un-damped system with two degrees of freedom as shown in Figure 6.1a, where the masses are constrained to move in the direction of the spring axis and executing free vibrations. The displacements are measured from the un-stretched positions of the springs. Let x_1 and x_2 be the displacement of the masses m_1 and m_2 respectively at any given instant of time measured from the

equilibrium position with $x_2 > x_1$. Then the spring forces acting on the masses are as shown in free body diagram in Figure 6.1b

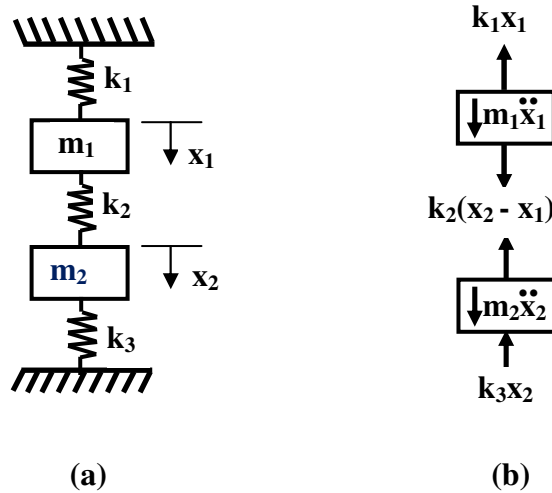


Figure 6.1

Based on Newton's second law of motion $\sum f = m \ddot{x}$

For mass m_1

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 = 0$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 = k_2 x_2 \quad \text{----- (1)}$$

for mass (2)

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + k_3 x_2 + k_2 x_2 - k_2 x_1 = 0$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 = k_2 x_1 \quad \text{----- (2)}$$

The solution for x_1 and x_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the mass m_1 execute harmonic vibration at frequency ω_1 and the mass m_2 execute harmonic vibration at frequency ω_2 then we have

$$x_1 = X_1 \sin \omega_1 t, \quad \text{and} \quad x_2 = X_2 \sin \omega_2 t \quad \text{----- (3)}$$

Where X_1 and X_2 are the amplitudes of vibrations of the two masses under steady state conditions.

Substituting equation (3) into equation (1) we have

$$-m_1 \omega_1^2 X_1 \sin \omega_1 t + (k_1 + k_2) X_1 \sin \omega_1 t = k_2 X_2 \sin \omega_2 t$$

$$\text{Therefore} \quad \frac{X_1}{X_2} = \frac{k_2 \sin \omega_2 t}{(k_1 + k_2) - m \omega_1^2 \sin \omega_1 t}$$

Since X_1 and X_2 are the amplitude of two harmonic motions, their ratio must be constant and independent of time. Therefore $\sin \omega_2 t / \sin \omega_1 t = C$ a constant.

Consider if $C > 1$. Then at time $t = \pi/2\omega_1$, $\sin\omega_1 t$ will be $\sin\omega_1 \times \pi/2\omega_1 = \sin \pi/2 = 1$

Therefore $\sin\omega_2 t / \sin\omega_1 t > 1$ or $\sin\omega_2 t > 1$ which is impossible. Hence $C > 1$ is not possible.

Similarly it can be shown that $C < 1$ is also not possible. Thus the only possibility is that $C = 1$

Hence $\sin\omega_2 t / \sin\omega_1 t = 1$ which is only possible if $\omega_2 = \omega_1 = \omega$. Hence the two harmonic motion have to be of the same frequency. Thus the solution of equation (1) and (2) can be

$$x_1 = X_1 \sin \omega t, \quad \text{and} \quad x_2 = X_2 \sin \omega t \quad \text{----- (4)}$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin \omega t \quad \ddot{x}_2 = -\omega^2 X_2 \sin \omega t \quad \text{----- (5)}$$

Substitute equation (4) and (5) into the equation (1) and (2)

$$-m_1 \omega^2 X_1 \sin \omega t + (k_1 + k_2) X_1 \sin \omega t = k_2 X_2 \sin \omega t \quad \text{----- (6)}$$

$$-m_2 \omega^2 X_2 \sin \omega t + (k_2 + k_3) X_2 \sin \omega t = k_2 X_1 \sin \omega t. \quad \text{----- (7)}$$

Canceling the common term $\sin \omega t$ on both the sides and re arranging the terms we have from equation (6)

$$X_1/X_2 = k_2 / (k_1 + k_2 - m_1 \omega^2) \quad \text{----- (8)}$$

$$X_1/X_2 = [(k_2 + k_3) - m_2 \omega^2] / k_2 \quad \text{----- (9)}$$

Thus equating equation (8) and (9) we have

$$X_1/X_2 = k_2 / (k_1 + k_2 - m_1 \omega^2) = [(k_2 + k_3) - m_2 \omega^2] / k_2 \quad \text{----- (10)}$$

Cross multiplying in equation (10) we have

$$(k_1 + k_2 - m_1 \omega^2) (k_2 + k_3 - m_2 \omega^2) = k_2^2 \quad \text{on simplification we get}$$

$$m_1 m_2 \omega^4 - [m_1 (k_2 + k_3) + m_2 (k_1 + k_2)] \omega^2 + [k_1 k_2 + k_2 k_3 + k_3 k_1] = 0 \quad \text{----- (11)}$$

The above equation (11) is quadratic in ω^2 and gives two values of ω^2 and therefore the two positive values of ω correspond to the two natural frequencies ω_{n1} and ω_{n2} of the system. The above equation is called frequency equation since the roots of the above equation give the natural frequencies of the system.

Now considering $m_1 = m_2 = m$ and $k_1 = k_3 = k$

Then the frequency equation (11) becomes

$$m^2 \omega^4 - 2m (k + k_2) \omega^2 + (k^2 + 2kk_2) = 0 \quad \text{----- (12)}$$

$$\text{Let: } \omega^2 = \lambda \quad \therefore \lambda^2 = \omega^4, \quad \therefore m^2 \lambda^2 - 2m (k + k_2) \lambda + (k^2 + 2kk_2) = 0 \quad \text{----- (13)}$$

The roots of the above equation (13) are as follows: Let $a = m^2$, $b = -2m (k + k_2)$; $c = (k^2 + 2kk_2)$

$$\therefore \lambda_{1,2} = [-b \pm \sqrt{b^2 - 4ac}] / 2a$$

$$\lambda_{1,2} = [-(-2m) (k + k_2) \pm \sqrt{[-2m (k+k_2)]^2 - 4 (m^2) (k^2 + 2kk_2)}] / 2m^2$$

$$= [+2m (k + k_2)] / 2m^2 \pm \sqrt{4m^2 [(k^2 + k_2^2 + 2kk_2) - (k^2 + 2kk_2)]} / 4m^4 = (k + k_2) / m \pm \sqrt{(k_2^2 / m^2)}$$

$$= (k + k_2) / m \pm k_2 / m$$

Thus $\lambda_1 = (k + k_2)/m - k_2 / m = k/m$. Then $\omega_{n1}^2 = K/m \therefore \omega_{n1} = \sqrt{(k/m)}$ ----- (14)

and $\therefore \lambda_2 = (k + 2k_2)/m$ Thus $\omega_{n2}^2 = (k + 2k_2)/m$. Then $\therefore \omega_{n2} = \sqrt{[(k + 2k_2)/m]}$ --- (15)

ω_{n1} is called the first or fundamental frequency or 1st mode frequency, ω_{n2} is called the second or 2nd mode frequency. Thus the number of natural frequencies of a system is equal to the number of degrees of freedom of system.

Modes Shapes: From equation (10) we have $X_1/X_2 = k_2/(k+k_2) - m\omega^2 = (k_2 + k) - m\omega^2/k_2$ ---(16)

Substitute $\omega_{n1} = \sqrt{(k/m)}$ in any one of the above equation (16).

$(X_1/X_2)_{\omega_{n1}} = k_2 / (k + k_2 - m(k/m))$ or $((k_2 + k) - m(k/m))/k_2 = k_2/k_2 = 1$

$(X_1/X_2)_{\omega_{n1}} = 1$ ----- (17)

Similarly substituting $\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$ in any one of the above equation (16).

$(X_1/X_2)_{\omega_{n2}} = k_2 / (k + k_2 - m(k+ 2k_2)/m)$ or $((k_2 + k) - m(k+ 2k_2)/m)/k_2 = - k_2/k_2 = -1$

$(X_1/X_2)_{\omega_{n2}} = -1$ ----- (18)

The displacements X_1 and X_2 corresponding to the two natural frequency of the system can be plotted as shown in Figure 6.2, which describe the mode in which the masses vibrate. When the system vibrates in principal mode the masses oscillate in such a manner that they reach maximum displacements simultaneously and pass through their equilibrium points simultaneously or all moving parts of the system oscillate in phase with one frequency. Since the ratio X_1/X_2 is important rather than the amplitudes themselves, it is customary to assign a unit value of amplitude to either X_1 or X_2 . When this is done, the principal mode is referred as normal mode of the system.

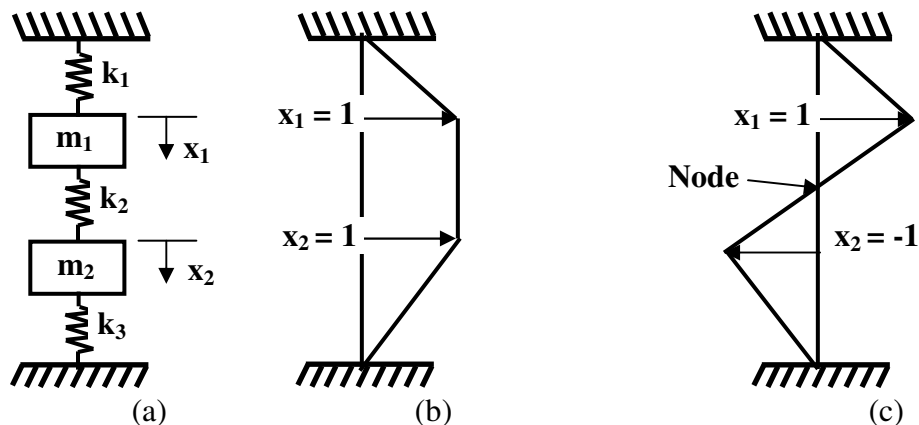


Figure – 6.2

$\omega_{n1} = \sqrt{(k/m)}$
1st Mode

$\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$
2nd Mode

It can be observed from the figure – 6.2b when the system vibrates at the first frequency, the amplitude of two masses remain same. The motion of both the masses are in phase i.e., both the masses move up or down together, the length of the middle spring remains constant, this spring (coupling spring) is neither stretched nor compressed. It moves rigid bodily with both the masses and hence totally ineffective as shown in Figure 6.3a. Even if the coupling spring is removed the two masses will vibrate as two single degree of freedom systems with $\omega_n = \sqrt{(K/m)}$.

When the system vibrates at the second frequency the displacement of the two masses have the same magnitude but with opposite signs. Thus the motions of m_1 and m_2 are 180° out of phase, the midpoint of the middle spring remains stationary for all the time. Such a point which experiences no vibratory motion is called a node, as shown in Figure 6.3b which is as if the middle of the coupling spring is fixed

When the two masses are given equal initial displacements in the same direction and released, they will vibrate at first frequency. When they are given equal initial displacements in opposite direction and released they will vibrate at the second frequency as shown in Figures 6.3a and 6.3b

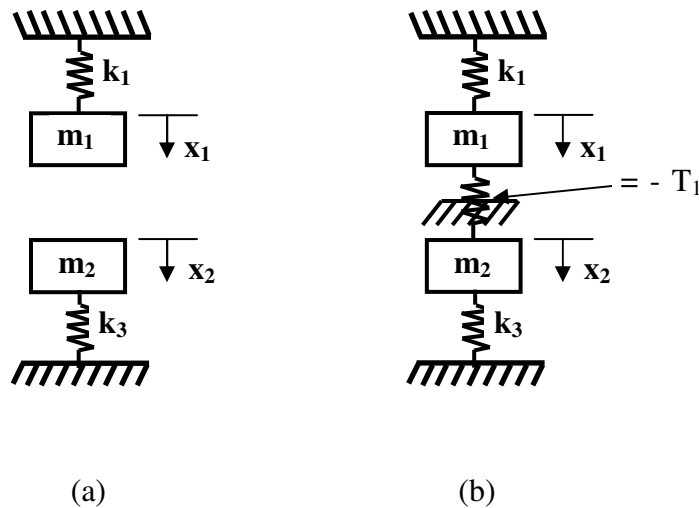


Figure – 6.3

$$\omega_{n1} = \sqrt{(k/m)}$$

1st Mode

$$\omega_{n2} = \sqrt{[(k + 2k_2)/m]}$$

2nd Mode

If unequal displacements are given to the masses in any direction, the motion will be superposition of two harmonic motions corresponding to the two natural frequencies.

1. Obtain the frequency equation for the system shown in Figure – 6.4. Also determine the natural frequencies and mode shapes when $k_1 = 2k$, $k_2 = k$, $m_1 = m$ and $m_2 = 2m$.

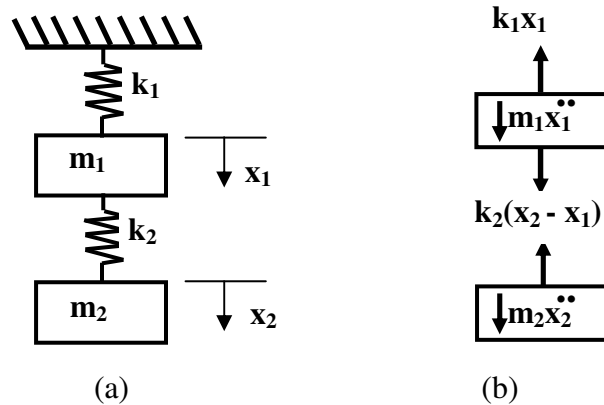


Figure – 6.4.

Solution

Consider two degrees of freedom system shown in Figure 6.4a, where the masses are constrained to move in the direction of the spring axis and executing free vibrations. The displacements are measured from the un-stretched positions of the springs. Let x_1 and x_2 be the displacement of the masses m_1 and m_2 respectively at any given instant of time measured from the equilibrium position with $x_2 > x_1$. Then the spring forces acting on the masses are as shown in free body diagram in Figure 6.4b

Based on Newton's second law of motion $\sum f = m \ddot{x}$

For mass m_1

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 = 0$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 = k_2 x_2 \quad \text{----- (1)}$$

for mass (2)

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$m_2 \ddot{x}_2 + k_2 x_2 = k_2 x_1 \quad \text{----- (2)}$$

The solution for x_1 and x_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the masses execute harmonic vibration at frequency ω . Thus if $x_1 = X_1 \sin \omega t$, and $x_2 = X_2 \sin \omega t$ ----- (3)

Then we have $x_1 = -\omega^2 X_1 \sin \omega t$, $x_2 = -\omega^2 X_2 \sin \omega t$ ----- (4)

Substitute equation (3) and (4) into the equation (1) and (2) we get

$$-m_1 \omega^2 X_1 \sin \omega t + (k_1 + k_2) X_1 \sin \omega t = k_2 X_2 \sin \omega t$$
 ----- (5)

$$-m_2 \omega^2 X_2 \sin \omega t + k_2 X_2 \sin \omega t = k_2 X_1 \sin \omega t$$
 ----- (6)

From equation (5) we have $X_1/X_2 = k_2 / [(k_1 + k_2) - m_1 \omega^2]$ ----- (7)

From equation (6) we have $X_1/X_2 = [k_2 - m_2 \omega^2] / k_2$ ----- (8)

Equating (7) and (8)

$$k_2 / (k_1 + k_2 - m_1 \omega^2) = [k_2 - m_2 \omega^2] / k_2$$

$$k_2^2 = (k_1 + k_2 - m_1 \omega^2) (k_2 - m_2 \omega^2)$$

$$k_2^2 = (k_1 + k_2) k_2 - m_1 \omega^2 k_2 - m_2 \omega^2 (k_1 + k_2) + m_1 m_2 \omega^4$$

$$m_1 m_2 \omega^4 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0$$
 ----- (9)

letting $\omega^2 = \lambda$ $m_1 m_2 \lambda^2 - \lambda [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0$ ----- (10)

Equation (10) is the frequency equation of the system which is quadratic in λ and hence the solution is

$$\lambda = [(m_1 k_2 + m_2 (k_1 + k_2)) \pm \sqrt{[(m_1 k_2 + m_2 (k_1 + k_2))]^2 - 4 m_1 m_2 k_1 k_2}] / 2 m_1 m_2$$

To determine the natural frequencies Given $k_1 = 2k$, $k_2 = k$ and $m_1 = m$, $m_2 = 2m$

$$\lambda = [mk + 2m(2k + k) \pm \sqrt{(mk + 6mk)^2 - 4m \cdot 2mk^2}] / 2m \cdot 2m$$

$$= [7mk \pm \sqrt{(7mk)^2 - 4(4m^2 k^2)}] / 4m^2$$

$$= [7mk \pm \sqrt{49m^2 k^2 - 16m^2 k^2}] / 4m^2$$

$$\lambda = [7mk \pm 5.744 mk] / 4m^2$$
 Thus $\lambda_1 = [7mk - 5.744 mk] / 4m^2$ and $\lambda_2 = [7mk + 5.744 mk] / 4m^2$

$$\lambda_1 = \omega_{n1}^2 = [7mk - 5.744 mk] / 4m^2 = 1.255 mk / 4m^2 = 0.3138 k/m$$
 Thus $\omega_{n1} = 0.56 \sqrt{(k/m)}$

$$\lambda_2 = \omega_{n2}^2 = [7mk + 5.744 mk] / 4m^2 = 3.186 k/m$$
 Thus $\omega_{n2} = 1.784 \sqrt{(k/m)}$

Substituting the values of frequencies into the amplitude ratio equation as given by equation (7) and (8) one can determine the mode shapes:

FOR THE FIRST MODE:

Substituting $\omega_{n1}^2 = 0.3138 K/m$ into either of the equation (7) or (8) we get first mode shape:

I.e. $X_1/X_2 = k_2 / [(k_1 + k_2) - m_1 \omega^2]$ ----- (7) $X_1/X_2 = [k_2 - m_2 \omega^2] / k_2$ ----- (8)

$$X_1/X_2 = k / [(2k + k) - m \omega^2] = k / [3k - m \cdot \omega_{n1}^2]$$

$$X_1/X_2 = k / [3k - 2m \cdot 0.3138k/m] = 1 / (3 - 0.3138) = 1/2.6862 = 0.3724$$

Thus we have $X_1/X_2 = 0.3724$. Then If $X_1 = 1$, $X_2 = 2.6852$

FOR THE SECOND MODE:

Substituting $\omega_{n2}^2 = 3.186 \text{ K/m}$ into either of the equation (7) or (8) we get first mode shape:

I.e. $X_1/X_2 = k_2 / [(k_1 + k_2) - m_1 \omega^2]$ ----- (7) $X_1/X_2 = [k_2 - m_2 \omega^2] / k_2$ ----- (8)

$X_1/X_2 = k / [(2k + k - m \omega^2)] = k / [3k - m \cdot \omega_{n1}^2]$

$X_1/X_2 = k / [3k - 2m \cdot 3.186 \text{ k/m}] = 1 / (3 - 3.186) = 1 / (-0.186) = -5.37 \text{ or}$

$X_1/X_2 = (k - 2m \cdot 3.186 \text{ k/m}) / k = [1 - 2(3.186)] = -5.37$

Thus we have $X_1/X_2 = -5.37$. Then If $X_1 = 1$, $X_2 = -0.186$

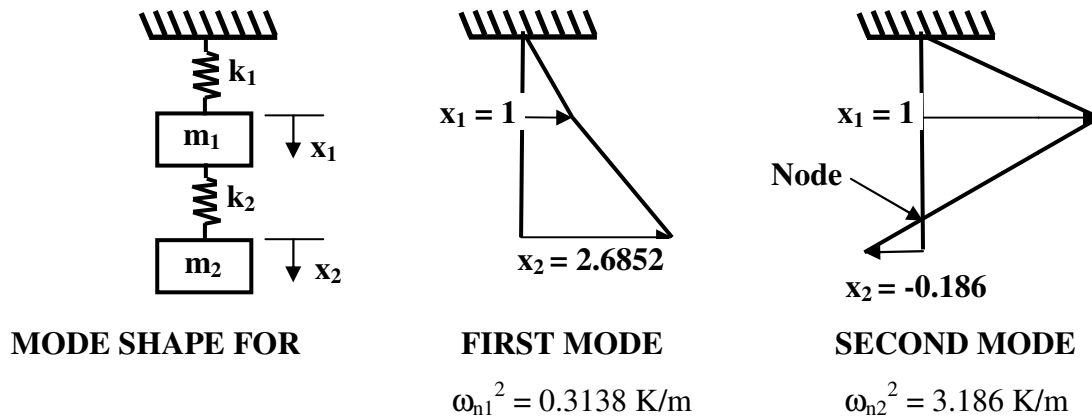


Figure – 6.5.

Derive the frequency equation for a double pendulum shown in figure.6.6. Determine the natural frequency and mode shapes of the double pendulum when $m_1 = m_2 = m$ $l_1 = l_2 = l$

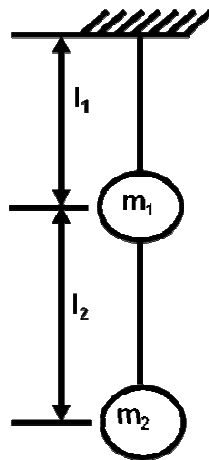


Figure 6.6

Consider two masses m_1 and m_2 suspended by string of length l_1 and l_2 as shown in the figure 6.6. Assume the system vibrates in vertical plane with small amplitude under which it only has the oscillation.

Let θ_1 and θ_2 be the angle at any given instant of time with the vertical and x_1 and x_2 be the horizontal displacement of the masses m_1 and m_2 from the initial vertical position respectively.

For small angular displacement we have $\sin\theta_1 = x_1 / l_1$ and $\sin\theta_2 = (x_2 - x_1) / l_2$ --- (1)

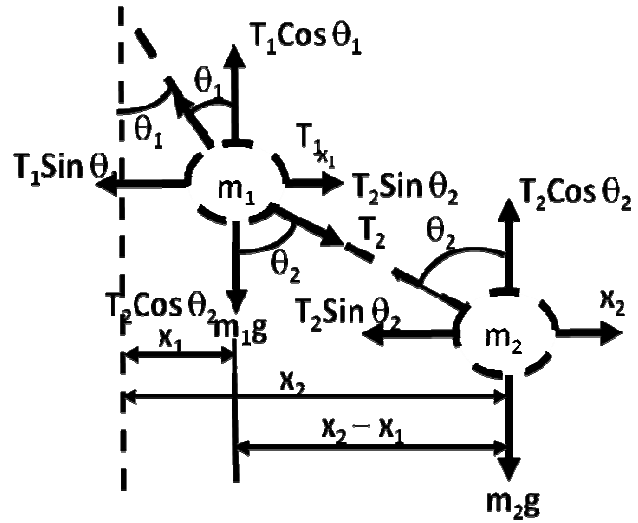


Figure.6.7 Free body diagram

Figure 6.7 shows the free body diagram for the two masses. For equilibrium under static condition the summation of the vertical forces should be equal to zero. Thus we have

At mass m_1 $T_1 \cos\theta_1 = m_1g + T_2 \cos\theta_2$ ----- (2)

At mass m_2 $T_2 \cos\theta_2 = m_2g$ ----- (3)

For smaller values of θ we have $\cos\theta = 1$. Then the above equations can be written as

$T_2 = m_2g$ ----- (4) and $T_1 = m_1g + m_2g$ $T_1 = (m_1 + m_2)g$ ----- (5)

When the system is in motion, the differential equation of motion in the horizontal direction can be derived by applying Newton Second Law of motion.

Then we have for mass m_1

$$m_1 \ddot{x}_1 = -T_1 \sin \theta_1 + T_2 \sin \theta_2$$

$$m_1 \ddot{x}_1 + T_1 \sin \theta_1 = T_2 \sin \theta_2 \quad \text{----- (6)}$$

For mass m_2

$$m_2 \ddot{x}_2 - T_2 \sin \theta_2 = 0$$

$$m_2 \ddot{x}_2 + T_2 \sin \theta_2 = 0 \quad \text{----- (7)}$$

Substituting the expression for T_2 and T_1 from equation (4) and (5) and for $\sin\theta_1$ and $\sin\theta_2$ from equation (1) into the above equation (6) and (7) we have

Equation (6) becomes

$$m_1 \ddot{x}_1 + [(m_1 + m_2) g](x_1/l_1) = m_2 g[(x_2 - x_1)/l_2]$$

$$m_1 \ddot{x}_1 + \{[(m_1 + m_2)/l_1] + m_2/l_2\}gx_1 = (m_2g/l_2) x_2 \quad \text{----- (8)}$$

Equation (7) becomes

$$m_2 \ddot{x}_2 + m_2 g(x_2 - x_1)/l_2$$

$$m_2 \ddot{x}_2 + (m_2 g/l_2) x_1 = (m_2 g/l_2) x_2 \text{ ----- (9)}$$

Equations (8) and (9) represent the governing differential equation of motion. Thus assuming the solution for the principal mode as

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \quad \text{and} \quad \ddot{x}_2 = -\omega^2 B \sin \omega t \text{ ----- (10)}$$

Substitute in (10) into equation (8) and (9) and cancelling the common term $\sin \omega t$ we have

$$[-m_1 \omega^2 + \{(m_1 + m_2)/l_1 + m_2/l_2\} g] A = (m_2 g/l_2) B \text{ --(11)}$$

$$[-m_2 \omega^2 + (m_2 g/l_2)] B = (m_2 g/l_2) A \text{ ----- (12)}$$

From equation (11) we have

$$A/B = (m_2 g/l_2) / [\{(m_1 + m_2)/l_1 + m_2/l_2\} g - m_1 \omega^2] \text{ -- (13)}$$

From equation (12) we have

$$A/B = [(m_2 g/l_2) - m_2 \omega^2] / (m_2 g/l_2) \text{ ----- (14)}$$

Equating equation (13) and (14) we have

$$A/B = (m_2 g/l_2) / [\{(m_1 + m_2)/l_1 + m_2/l_2\} g - m_1 \omega^2] = [(m_2 g/l_2) - m_2 \omega^2] / (m_2 g/l_2)$$

$$[\{(m_1 + m_2)/l_1 + m_2/l_2\} g - m_1 \omega^2] [(m_2 g/l_2) - m_2 \omega^2] = (m_2 g/l_2)^2 \text{ ----- (15)}$$

Equation (15) is a the quadratic equation in ω^2 which is known as the frequency equation.

Solving for ω^2 we get the natural frequency of the system.

Particular Case:

When $m_1 = m_2 = m$ and $l_1 = l_2 = l$

Then equation (13) will be written as $A/B = (mg/l) / [(3mg/l) - m\omega^2] = (g/l) / [(3g/l) - \omega^2]$

$$A/B = 1 / [3 - (\omega^2 l/g)] \text{ ----- (16)}$$

and equation (14) will be written as $A/B = [1 - (\omega^2 l/g)] \text{ ----- (17)}$

Equating equation (16) and (17) we get $A/B = 1 / [3 - (\omega^2 l/g)] = [1 - (\omega^2 l/g)]$

$$[3 - (\omega^2 l/g)] * [1 - (\omega^2 l/g)] = 1 \quad \text{or} \quad (3g - \omega^2 l) * (g - \omega^2 l) = g^2$$

$$3g^2 - 3gl\omega^2 - gl\omega^2 + l^2\omega^4 = g^2 \quad \text{or} \quad l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0 \quad \text{or}$$

$$\omega^4 - (4g/l)\omega^2 + (2g^2/l^2) = 0 \text{ ----- (18)}$$

letting $\lambda = \omega^2$ in equation (18) we get $\lambda^2 - (4g/l)\lambda + (2g^2/l^2) = 0 \text{ ----- (19)}$

Which is a quadratic equation in λ and the solution for the equation (19) is

$$\lambda_{1,2} = (2g/l) \pm \sqrt{[(4g^2/l^2) - (2g^2/l^2)]} \quad \text{or} \quad \lambda_{1,2} = (g/l)(2 \pm \sqrt{2}) \quad \text{----- (20)}$$

$$\lambda_1 = (g/l)(2 - \sqrt{2}) = 0.5858(g/l) \quad \text{----- (21)} \quad \text{and} \quad \lambda_2 = (g/l)(2 + \sqrt{2}) = 3.4142(g/l) \quad \text{----- (22)}$$

Since $\lambda = \omega^2$ then the natural frequency $\omega_{n1} = \sqrt{\lambda_1} = 0.7654\sqrt{(g/l)}$ thus $\omega_{n1} = 0.7654\sqrt{(g/l)}$ --- (23)

and $\omega_{n2} = \sqrt{\lambda_2} = 1.8478\sqrt{(g/l)}$ thus $\omega_{n2} = 1.8478\sqrt{(g/l)}$ ----- (24)

Substituting ω_{n1} and ω_{n2} from equation (23) and (24) into either of the equation (16) or (17) we get the mode shape

FOR THE FIRST MODE:

Mode shapes for the first natural frequency $\omega_{n1} = 0.7654\sqrt{(g/l)}$ or $\omega_{n1}^2 = (g/l)(2 - \sqrt{2})$

I mode from equation (16) $A/B = 1/[3 - (\omega^2 l/g)]$

$$(A/B)_1 = 1/[3 - \omega_{n1}^2 l/g] = 1/[3 - \{(g/l)(2 - \sqrt{2})\} l/g] = 1/(3 - 2 + \sqrt{2}) = 1/(1 + \sqrt{2}) = 1/2.4142 = 0.4142$$

Thus when $A = 1$ $B = 2.4142$

Also from equation (17) $A/B = [1 - (\omega^2 l/g)]$

For $\omega_{n1} = 0.7654\sqrt{(g/l)}$ or $\omega_{n1}^2 = (g/l)(2 - \sqrt{2})$

$$(A/B)_1 = (1 - \omega_{n1}^2 l/g) = [1 - \{(g/l)(2 - \sqrt{2})\} l/g]$$

$$\text{or} \quad (A/B)_1 = (1 - 2 + \sqrt{2}) = \sqrt{2} - 1 = 0.4142$$

Thus when $A = 1$ $B = 2.4142$

Modes shape is shown in figure-6.8

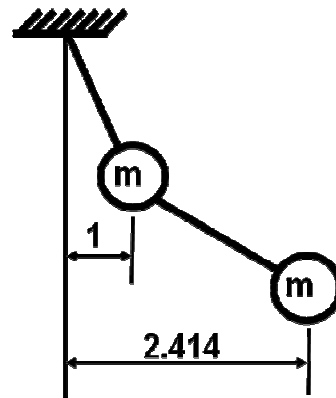


Figure- 6.8

FOR THE SECOND MODE:

Mode shapes for second natural frequency $\omega_{n2} = 1.8478\sqrt{(g/l)}$ or $\omega_{n2}^2 = (g/l)(2 + \sqrt{2})$

II mode from equation (16) is given by $A/B = 1/[3 - (\omega^2 l/g)]$

$$(A/B)_2 = 1/[3 - \omega_{n2}^2 l/g] = 1/[3 - \{(g/l)(2 + \sqrt{2})\} l/g] = 1/(3 - 2 - \sqrt{2}) = 1/(-1 - \sqrt{2})$$

$$(A/B)_2 = 1/(-0.4142) = -2.4142 \quad \text{or} \quad \text{Thus when } A = 1 \text{ } B = -0.4142$$

Also from equation (17) $A/B = [1 - (\omega^2 l/g)]$

$\omega_{n2} = 1.8478\sqrt{(g/l)}$ or $\omega_{n2}^2 = (g/l)(2 + \sqrt{2})$

$$(A/B)_2 = (1 - \omega_{n2}^2 l/g) = [1 - \{(g/l)(2 + \sqrt{2})\} l/g]$$

$$(A/B)_2 = (1 - 2 - \sqrt{2}) = -(1 + \sqrt{2}) = -2.4142$$

Thus when $A = 1$ $B = -0.4142$

Modes shape is shown in figure-6.9

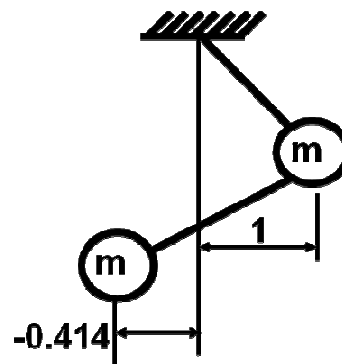


Figure- 6.9

Determine the natural frequencies of the coupled pendulum shown in the figure – 6.10. Assume that the light spring of stiffness ‘k’ is un-stretched and the pendulums are vertical in the equilibrium position.

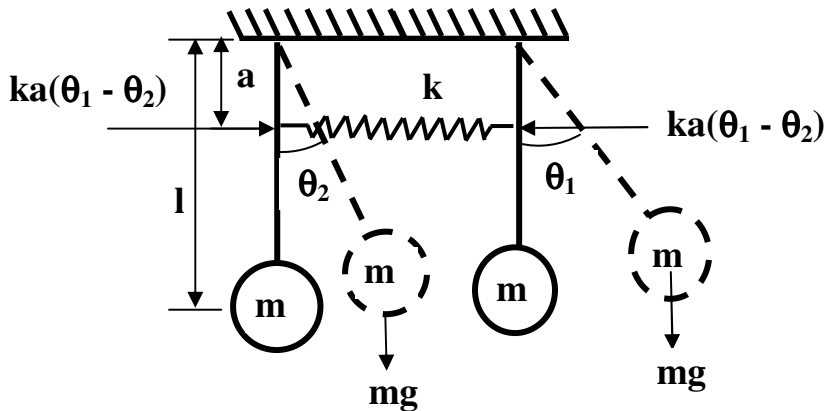


Figure – 6.10.

Solution:

Considering counter clockwise angular displacement to be positive and taking the moments about the pivotal point of suspension by D.Alembert’s principle we have

$$ml^2 \ddot{\theta}_1 = - mgl\theta_1 - ka(\theta_1 - \theta_2) \quad \text{----- (1)}$$

$$ml^2 \ddot{\theta}_2 = - mgl\theta_2 + ka(\theta_1 - \theta_2) \quad \text{----- (2)}$$

Equation (1) and (2) can also be written as

$$ml^2 \ddot{\theta}_1 + (mgl + ka)\theta_1 = ka\theta_2 \quad \text{----- (3)}$$

$$ml^2 \ddot{\theta}_2 + (mgl + ka)\theta_2 = ka\theta_1 \quad \text{----- (4)}$$

Equation (3) and (4) are the second order differential equation and the solution for θ_1 and θ_2 are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the masses execute harmonic vibration at frequency ω

Thus if $\theta_1 = A \sin \omega t$, and $\theta_2 = B \sin \omega t$ ----- (5)

Substitute equation (5) into the equation (3) and (4) and canceling the common terms we get

$$(- ml^2\omega^2 + mgl + ka^2)A = ka^2B \quad \text{----- (6)}$$

$$(- ml^2\omega^2 + mgl + ka^2)B = ka^2A \quad \text{----- (7)}$$

From equation (6) we have $A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2]$ ----- (8)

From equation (7) we have $A/B = [mgl + ka^2 - ml^2\omega^2] / ka^2$ ----- (9)

Equating (8) and (9)

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = [mgl + ka^2 - ml^2\omega^2] / ka^2$$

$$[mgl + ka^2 - ml^2\omega^2]^2 = [ka^2]^2 \quad \text{----- (10) or}$$

$$mgl + ka^2 - ml^2\omega^2 = \pm ka^2 \quad \omega^2 = (mgl + ka^2 \pm ka^2) / ml^2 \quad \text{----- (11)}$$

$$\omega_{1,2} = \sqrt{[(mgl + ka^2 \pm ka^2) / ml^2]} \text{ ----- (12)}$$

$$\omega_1 = \sqrt{[(mgl + ka^2 - ka^2) / ml^2]} = \sqrt{(g/l)} \text{ ----- (13)}$$

$$\omega_2 = \sqrt{[(mgl + ka^2 + ka^2) / ml^2]} = \sqrt{[(g/l) + (2ka^2/ml^2)]} \text{ ----- (14)}$$

Substituting the values of frequencies into the amplitude ratio equation as given by equation (8) and (9) one can determine the mode shapes:

FOR THE FIRST MODE:

Substituting $\omega_{n1}^2 = g/l$ into either of the equation (8) or (9) we get first mode shape:

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = ka^2 / [mgl + ka^2 - ml^2g/l] = ka^2 / [mgl + ka^2 - mlg] = ka^2 / ka^2$$

$$A/B = 1$$

FOR THE SECOND MODE:

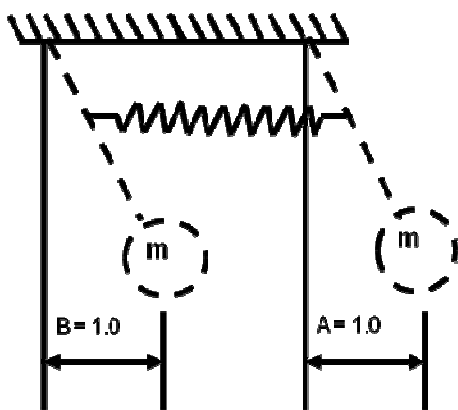
Substituting $\omega_{n2}^2 = [(g/l) + (2ka^2/ml^2)]$ into either of the equation (8) or (9) we get second mode shape:

$$A/B = ka^2 / [mgl + ka^2 - ml^2\omega^2] = ka^2 / [mgl + ka^2 - ml^2[(g/l) + (2ka^2/ml^2)]]$$

$$= ka^2 / [mgl + ka^2 - mlg - 2ka^2] = (ka^2 / -ka^2) = -1 \text{ Thus } A/B = -1$$

Mode shapes at these two natural frequencies are as shown in figure- 6.10

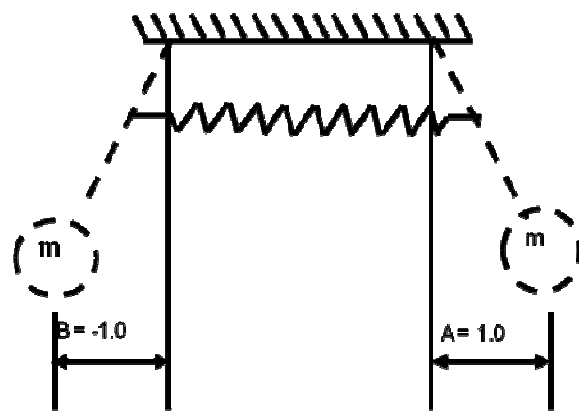
MODE SHAPES AT TWO DIFFERENT FREQUENCIES



FIRST MODE

$$\omega_{n1}^2 = g/l \quad A/B = 1$$

Figure-6.10 Mode Shapes at first frequency



SECOND MODE

$$\omega_{n2}^2 = [(g/l) + (2ka^2/ml^2)] \quad A/B = -1$$

Figure-6.11 Mode Shapes at second frequency

Derive the equation of motion of the system shown in figure 6.12. Assume that the initial tension 'T' in the string is too large and remains constants for small amplitudes. Determine the natural frequencies, the ratio of amplitudes and locate the nodes for each mode of vibrations when $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = 1$.

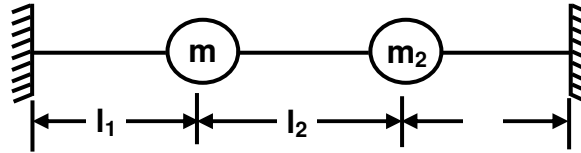


Figure 6.12.

At any given instant of time let y_1 and y_2 be the displacement of the two masses m_1 and m_2 respectively. The configuration is as shown in the figure 6.13.

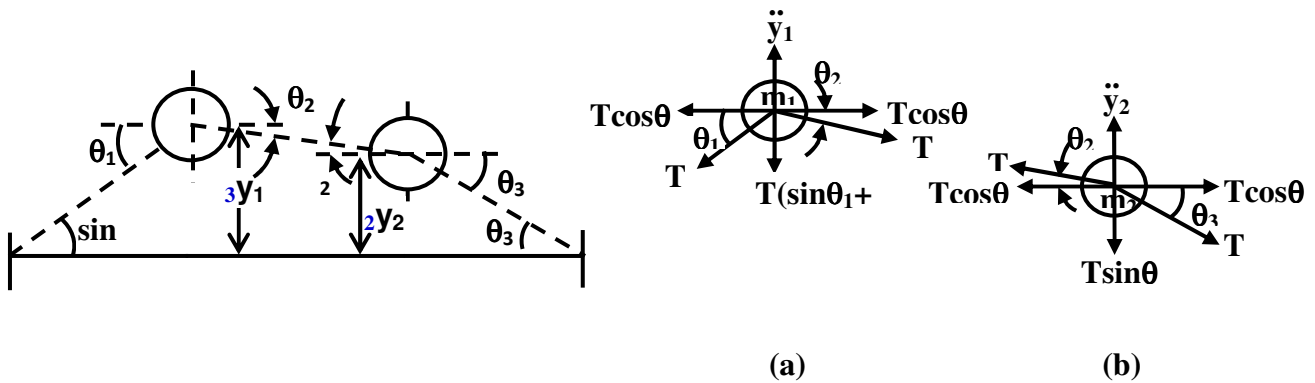


Figure 6.13.

Figure 6.14

The forces acting on the two masses are shown in the free body diagram in figure 6.14(a) and (b)

From figure 6.13 we have $\sin\theta_1 = (y_1/l_1)$ $\sin\theta_2 = [(y_1 - y_2)/l_2]$ and $\sin\theta_3 = (y_2/l_3)$

For small angle we have $\sin\theta_1 = \theta_1 = (y_1/l_1)$, $\sin\theta_2 = \theta_2 = [(y_1 - y_2)/l_2]$ and $\sin\theta_3 = \theta_3 = (y_2/l_3)$

and $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = 1.0$ Thus the equation of motion for lateral movement of the masses

For the mass m_1

$$m_1 \ddot{y}_1 = - (T \sin\theta_1 + T \sin\theta_2) = - T (\theta_1 + \theta_2)$$

$$m_1 \ddot{y}_1 = - T [(y_1/l_1) + (y_1 - y_2)/l_2] \quad \text{or} \quad m_1 \ddot{y}_1 + [(T/l_1) + (T/l_2)]y_1 = (T/l_2)y_2 \quad \text{---- (1)}$$

For the mass m_2

$$m_2 \ddot{y}_2 = (T \sin\theta_2 - T \sin\theta_3)$$

$$m_2 \ddot{y}_2 = T [(y_1 - y_2)/l_2 - (y_2/l_3)] \quad \text{or} \quad m_2 \ddot{y}_2 + [(T/l_2) + (T/l_3)]y_2 = (T/l_2)y_1 \quad \text{----- (2)}$$

Assuming harmonic motion as $y_1 = A \sin \omega t$ and $y_2 = B \sin \omega t$ ----- (3) and substituting this

into equation (1) and (2) we have $[-m_1 \omega^2 + (T/l_1) + (T/l_2)] A = (T/l_2) B$ ----- (4)

$[-m_2 \omega^2 + (T/l_2) + (T/l_3)] B = (T/l_2) A$ ----- (5)

Thus from equation (4) we have $A/B = (T/l_2) / [(T/l_1) + (T/l_2) - m_1 \omega^2]$ ----- (6)

and from equation (5) we have $A/B = [(T/l_2) + (T/l_3) - m_2 \omega^2] / (T/l_2)$ ----- (7)

Equating equation (6) and (7) we have $A/B = (T/l_2) / [(T/l_1) + (T/l_2) - m_1 \omega^2] \cdot [(T/l_2) + (T/l_3) - m_2 \omega^2] / (T/l_2)$

Thus we have $[(T/l_1) + (T/l_2) - m_1 \omega^2][(T/l_2) + (T/l_3) - m_2 \omega^2] = (T^2/l_2^2)$ ----- (8)

Equation (8) is the equation on motion which is also known as frequency equation. Solving this equation gives the natural frequencies of the system.

Particular Case: When $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$ then equation (6) can be written as

$A/B = (T/l) / [(T/l) + (T/l) - m \omega^2] = (T/l) / [(2T/l) - m \omega^2]$ ----- (9)

and equation (7) can be written as $A/B = [(T/l) + (T/l) - m \omega^2] / (T/l) = [(2T/l) - m \omega^2] / (T/l)$ ----- (10)

Equating equation (9) and (10) we have $[(2T/l) - m \omega^2]^2 = (T/l)^2$ ----- (11)

Thus $2T/l - m \omega^2 = \pm (T/l)$ ----- (12) Therefore we have $\omega^2 = [(2T \pm T)/ml]$ ----- (13)

$\omega_{n1} = \sqrt{[(2T-T)/ml]} = \sqrt{(T/ml)}$ ----- (14) and $\omega_{n2} = \sqrt{[(2T+T)/ml]} = \sqrt{(3T/ml)}$ ----- (15)

Substituting equation (14) and (15) into either of the equation (9) or (10) we have the ratio of amplitudes for the two natural frequencies. For the first natural frequency $\omega_{n1} = \sqrt{(T/ml)}$ then from

equation (9) we have $(A/B)_{\omega_{n1}} = (T/l) / [(2T/l) - m \omega^2] = (T/l) / [(2T/l) - m(T/ml)] = (T/l) / (T/l) = +1$

or from equation (10) we have $(A/B)_{\omega_{n1}} = [(2T/l) - m \omega^2] / (T/l) = [(2T/l) - m(T/ml)] / (T/l)$

Thus $(A/B)_{\omega_{n1}} = (T/l) / (T/l) = +1$

For the second natural frequency $\omega_{n2} = \sqrt{(3T/ml)}$ then from equation (9) we have

$(A/B)_{\omega_{n2}} = (T/l) / [(2T/l) - m \omega^2] = (T/l) / [(2T/l) - m(3T/ml)] = (T/l) / (-T/l) = -1$

Thus $(A/B)_{\omega_{n2}} = (T/l) / (-T/l) = -1$ Then the mode shape will be as shown in figure 6.15(a) and (b)

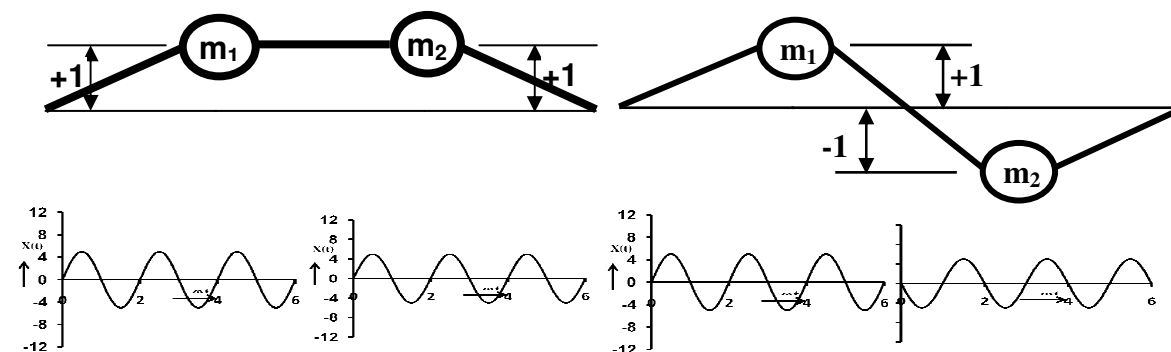


Figure 6.15(a)

First Mode $\omega_{n1} = \sqrt{(T/ml)}$, $(A/B)_{\omega_{n1}} = +1$

Figure 6.15(b)

Second Mode $\omega_{n2} = \sqrt{(3T/ml)}$, $(A/B)_{\omega_{n1}} = -1$

Derive the equation of motion of a torsional system shown in figure 6.16. Let J_1 and J_2 be the mass moment of inertia of the two rotors which are coupled by shafts having torsional stiffness of K_{t1} and K_{t2} .

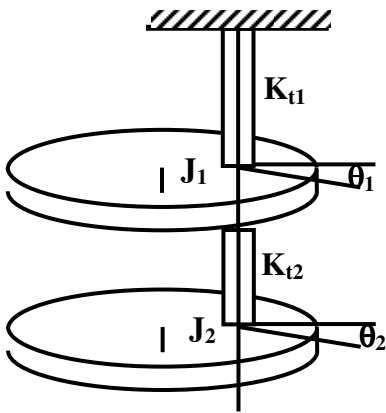


Figure 6.16 Two Degree of Freedom torsional system

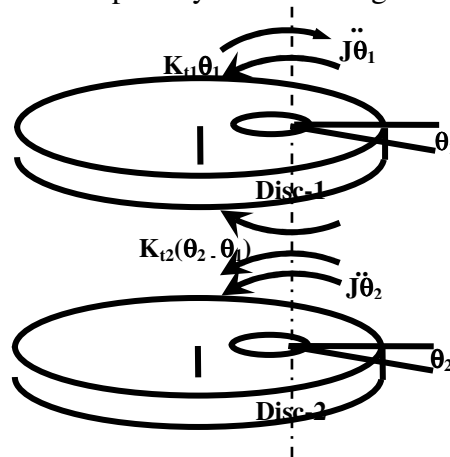


Figure 6.17 Free Body Diagram

If θ_1 and θ_2 are the angular displacement of the two rotors at any given instant of time, then the shaft with the torsional stiffness K_{t1} exerts a torque of $K_{t1}\theta_1$ and the shaft with the torsional stiffness K_{t2} exerts a torque of $K_{t2}(\theta_2 - \theta_1)$ as shown in the free body diagram figure 6.17

Then by Newton second law of motion we have for the mass m_1

$$J_1\ddot{\theta}_1 = -K_{t1}\theta_1 + K_{t2}(\theta_2 - \theta_1) \quad \text{or} \quad J_1\ddot{\theta}_1 + (K_{t1} + K_{t2})\theta_1 = K_{t2}\theta_2 \quad \text{----- (1)}$$

for the mass m_2

$$J_2\ddot{\theta}_2 = -K_{t2}(\theta_2 - \theta_1) \quad \text{or} \quad J_2\ddot{\theta}_2 + K_{t2}\theta_2 = K_{t2}\theta_1 \quad \text{----- (2)}$$

Equation (1) and (2) are the governing Equations of motion of the system.

Equivalent Shaft for a Torsional system

In many engineering applications we find shaft of different diameters as shown in Figure 6.18 are in use.

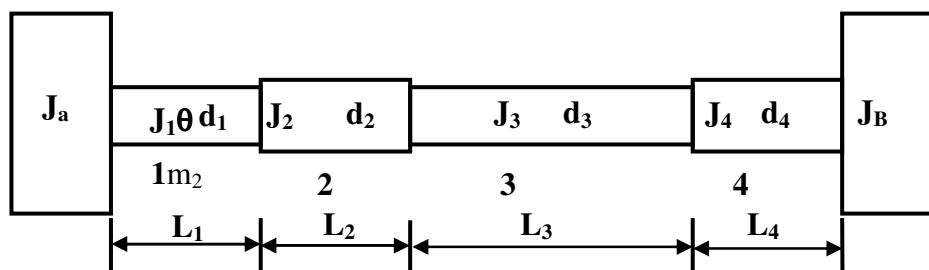


Figure-6.18 Stepped shaft

For vibration analysis it is required to have an equivalent system. In this section we will study how to obtain the torsionally equivalent shaft. Let θ be the total angle of twist in the shaft by application of torque T , and $\theta_1, \theta_2, \theta_3$ and θ_4 be twists in section 1, 2, 3 and 4 respectively. Then we have

$$\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$$

From torsion theory we have,

$$\frac{T}{J} = \frac{G\theta}{L} \quad \text{Where } J = \frac{\pi d^4}{32} \text{ Polar moment of inertia of shaft.}$$

Thus $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$ will be

$$\theta = \frac{TL_1}{J_1G_1} + \frac{TL_2}{J_2G_2} + \frac{TL_3}{J_3G_3} + \frac{TL_4}{J_4G_4}$$

If material of shaft is same, then the above equation can be written as

$$\theta = \frac{32T}{\pi G} \left[\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4} + \frac{L_3}{d_3^4} + \frac{L_4}{d_4^4} \right]$$

If d_e and L_e are equivalent diameter and lengths of the shaft, then:

$$\frac{L_e}{d_e^4} = \left[\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4} + \frac{L_3}{d_3^4} + \frac{L_4}{d_4^4} \right]$$

$$L_e = L_1 \left[\frac{d_e}{d_1} \right]^4 + L_2 \left[\frac{d_e}{d_2} \right]^4 + L_3 \left[\frac{d_e}{d_3} \right]^4 + L_4 \left[\frac{d_e}{d_4} \right]^4$$

Equivalent shaft of the system shown in Figure- 6.19

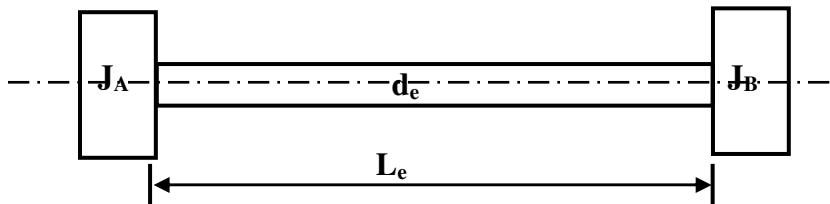


Figure – 6.19 Equivalent shaft of the system shown in figure – 6.18

Definite and Semi-Definite Systems

Definite Systems

A system, which is fixed from one end or both the ends is referred as definite system. A definite system has nonzero lower natural frequency. A system, which is free from both the ends, is referred as semi-definite system. For semi-definite systems, the first natural frequency is zero.

Various definite linear and a torsional systems are shown in figure-6.19

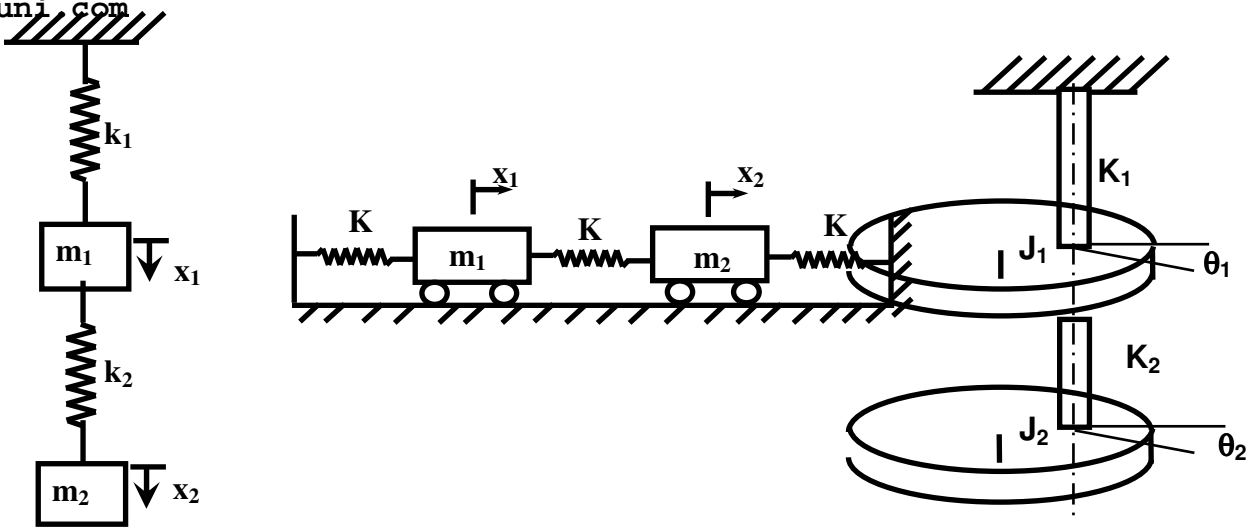


Figure-6.19 Various definite systems

Semi Definite or Degenerate System Systems

Systems for which one of the natural frequencies is equal to zero are called semi definite systems.

Various definite linear and a torsional systems are shown in figure-6.20

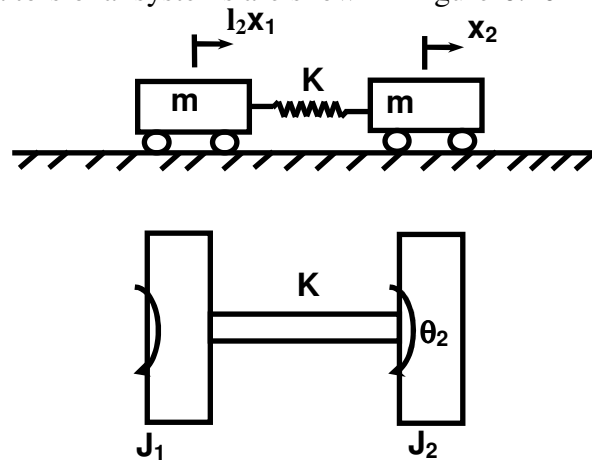


Figure-6.20 Semi-Definite systems

Problem to solve

Derive the equation of motion of a torsional system shown in figure 6.21.

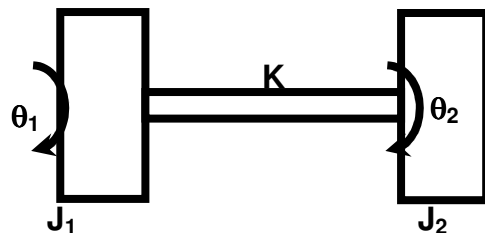


Figure-6.21 Two Rotor System

Vibration of Geared Systems

Consider a Turbo-generator geared system is shown in the figure 6.22.

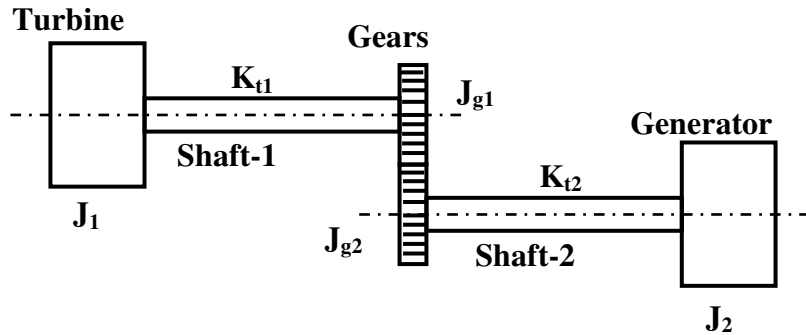


Figure-6.22: Turbo-Generator Geared System.

The analysis of this system is complex due to the presence of gears. Let 'i' be the speed ratio of the system given by

$$i = \frac{\text{Speed of Turbine}}{\text{Speed of Generator}}$$

First step in the analysis of this system is to convert the original geared system into an equivalent rotor system. Which is done with respect to either of the shafts.

When the Inertia of Gears is Neglected

The basis for this conversion is to consider the energies i.e. the kinetic and potential energy for the equivalent system should be same as that of the original system. Thus if θ_1 and θ_2 are the angular displacement of the rotors of moment of inertia J_1 and J_2 respectively then neglecting the inertia of the gears the **Kinetic and Potential energy of the original system are given by**

$$T = 1/2 J_1 \dot{\theta}_1^2 + 1/2 J_2 \dot{\theta}_2^2 \quad U = 1/2 k_{t1} \theta_1^2 + 1/2 k_{t2} \theta_2^2$$

Since $\theta_2 = i\theta_1$ Then the above equations can be written as

$$T = 1/2 J_1 \dot{\theta}_1^2 + 1/2 J_2 (i\dot{\theta}_1)^2 = 1/2 J_1 \dot{\theta}_1^2 + 1/2 (i^2 J_2) \dot{\theta}_1^2$$

$$U = 1/2 k_{t1} \theta_1^2 + 1/2 k_{t2} (i\theta_1)^2 = 1/2 k_{t1} \theta_1^2 + 1/2 (i^2 k_{t2}) \theta_1^2$$

Thus the above equation shows that the original system can be converted into equivalent system with respect to the first shaft as shown in figure- 6.23

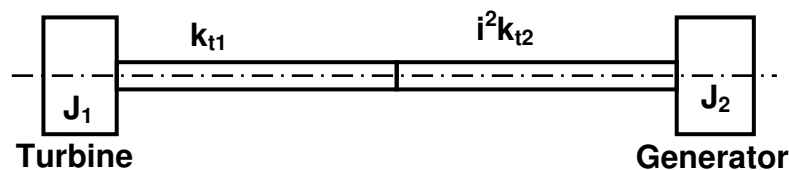


Figure-6.23 Turbo-generator geared system neglecting the inertia of gears

Which is obtained by multiplying the inertia and stiffness of the second shaft by i^2 and keeping this part of the system in series with the first part. Thus the stiffness of this equivalent two rotor system is

$$k_{te} = i^2 k_{t1}k_{t2} / (k_{t1} + i^2 k_{t2})$$

Thus the frequency of the system is given by $\omega_n = \sqrt{k_{te}(J_1 + i^2 J_2) / i^2 J_1 J_2}$ rad/sec

When the Inertia of Gears is Considered

If the inertia of the gears is not negligible then the equivalent system with respect to the first shaft can be obtained in the same manner and finally we have the three rotor system as shown in figure-6.24

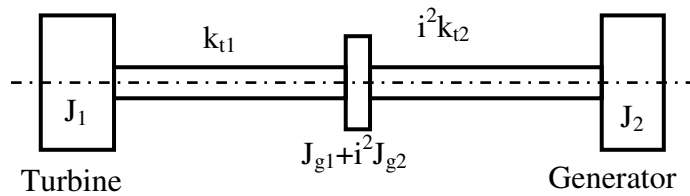


Figure-6.24 Considering the inertia of gears

CO-ORDINATE COUPLING AND PRINCIPAL COORDINATES.

Consider a two degree of freedom system as shown in the figure- 6.25. The vibration is restricted in plane of paper.

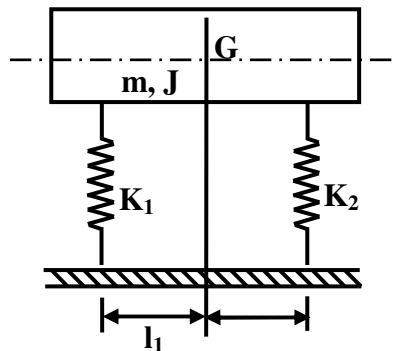


Figure-6.25. Two degree of freedom system

If m is the mass of the system, J is the Mass Moment of Inertia the system and G is the centre of gravity. k1 and k2 are the stiffness of the springs which are at a distance 'l1' and 'l2' from the line passing through the centre of gravity of the mass.

Then the system has two generalized co-ordinates, x is in Cartesian and θ is in Polar co-ordinate systems when it is vibrating.

At any given instant of time for a small disturbance the system occupy the position as shown in figure-6.26(a).

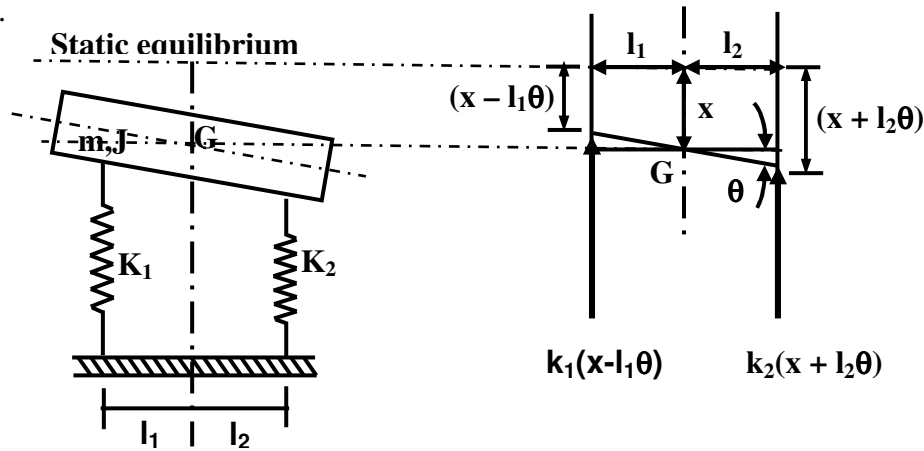


Figure-6.26 (a) system under vibration (b) displacements at the springs

If 'x' is the displacement at the center of gravity of the system. Then the amount of displacements that take place at the left spring is $(x - l_1\theta)$ and at the right spring is $(x + l_2\theta)$ which is as shown in figure-6.26(b).

At any given instant of time when the body is displaced through a rectilinear displacement 'x' and an angular displacement 'θ' from its equilibrium position. The left spring with the stiffness k_1 and the right spring with the stiffness k_2 are compressed through $(x - l_1\theta)$ and $(x + l_2\theta)$ from their equilibrium position, The forces acting on the system is as shown in the free body diagram in figure-6.26. The differential equation of motion of the system in 'x' and 'θ' direction are written by considering the forces and moments in their respective direction.

Thus we have the equation of motion

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \text{ ----- (1)}$$

$$J\ddot{\theta} = +k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2 \text{ ----- (2)}$$

Rearranging the above two equation we have

$$m\ddot{x} + (k_1 + k_2)x = (k_1l_1 - k_2l_2)\theta \text{ ----- (3)}$$

$$J\ddot{\theta} + (k_1l_1^2 + k_2l_2^2)\theta = (k_1l_1 - k_2l_2)x \text{ ----- (4)}$$

Since $J = mr^2$ The above two equation can also be written as

$$\ddot{x} + [(k_1 + k_2)/m]x = [(k_1l_1 - k_2l_2)/m]\theta \text{ ----- (5)}$$

$$\ddot{\theta} + [(k_1l_1^2 + k_2l_2^2)/mr^2]\theta = [(k_1l_1 - k_2l_2)/mr^2]x \text{ ----- (6)}$$

Letting $[(k_1 + k_2)/m = a, (k_1l_1 - k_2l_2)/m = b$ and $(k_1l_1^2 + k_2l_2^2)/mr^2 = c$

Thus substituting these into equation (5) and (6) we have

$$\ddot{x} + ax = b\theta \text{ ----- (7)}$$

$$\ddot{\theta} + c\theta = (b/r^2)x \text{ ----- (8)}$$

The above two differential equation (7) and (8) are coupled with respect to the coordinates in which 'b' is called the coupling coefficient or coordinate coupling.

Since if b=0 the two coordinate coupling equations (7) and (8) are independent of each other. The two equations are then decoupled and each equation may be solved independently of the other. Such a coordinate are called **PRINCIPAL COORDINATE OR NORMAL COORDINATES**.

Therefore the two i.e. rectilinear and angular motions can exists independently of each other with their natural frequency as \sqrt{a} and \sqrt{c} .

Thus for the case of decoupled system when b=0 then $(k_1l_1 - k_2l_2)/m = 0$ or $k_1l_1 - k_2l_2 = 0$ or $k_1l_1 = k_2l_2$. Then the natural in rectilinear and angular modes are $\omega_{n1} = \sqrt{a}$ and $\omega_{na} = \sqrt{c}$

$$\omega_{n1} = \sqrt{a} = \sqrt{(k_1 + k_2)/m} \text{ and } \omega_{na} = \sqrt{c} = \sqrt{(k_1l_1^2 + k_2l_2^2)/mr^2}$$

In general for a two degree of freedom under damped free vibration the equation of motion can be written in the matrix form as

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Which reveal the type of coupling present in the system as Dynamic or Mass Coupling exist if the mass matrix is non diagonal matrix. Where as stiffness or static Coupling exist if the stiffness matrix is non diagonal. Where as damping Coupling exist if the damping matrix is non diagonal.

The system is dynamically decoupled when the mass matrix exists is a diagonal matrix.

$$\begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The system is damped decoupled when the damping matrix exists is a diagonal matrix.

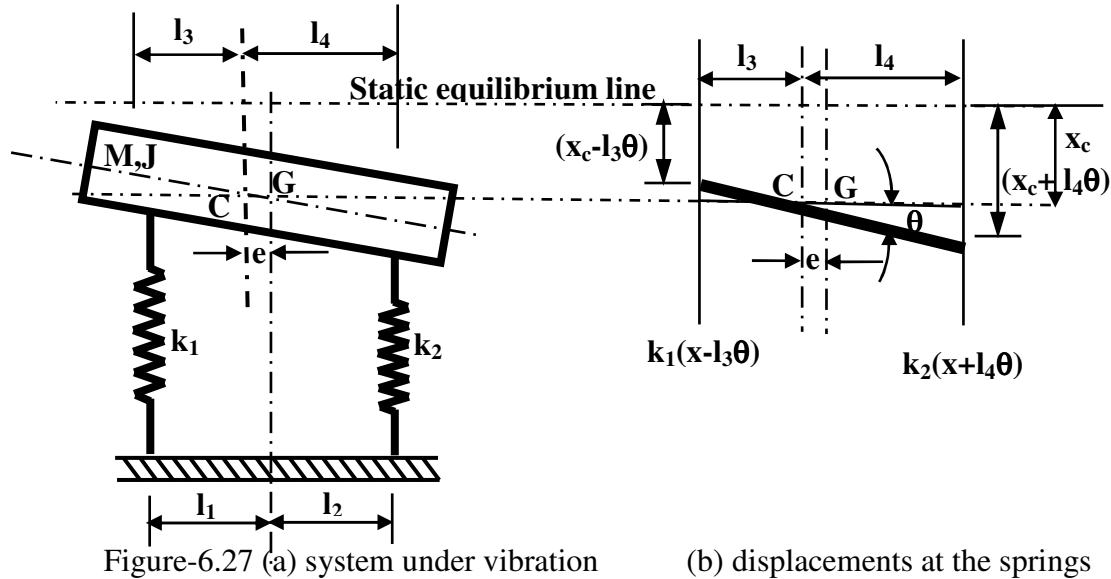
$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The system is statically decoupled when the stiffness matrix exists is a diagonal matrix.

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Dynamic or Mass Coupling:

If there is some point 'C' in the system along which a force is applied to the system produces pure translation along the line of action of force as shown in figure-6.27



Then the equation of motion is

$$M\ddot{x}_c + me\ddot{\theta} = -k_1(x_c - l_3\theta) - k_2(x_c + l_4\theta)$$

$$J\ddot{\theta} + me\ddot{x}_c = -k_1(x_c - l_3\theta) - k_2(x_c + l_4\theta)$$

Rearranging the above two equation we have

$$M\ddot{x}_c + me\ddot{\theta} + (k_1 + k_2)x_c + (k_2l_4 - k_1l_3)\theta = 0$$

$$J\ddot{\theta} + me\ddot{x}_c + (k_2l_4 - k_1l_3)x_c + (k_1l_3^2 + k_2l_4^2)\theta = 0$$

The above equation can be written in matrix form as

$$\begin{pmatrix} M & me \\ me & J \end{pmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{pmatrix} (k_1 + k_2) & (k_2l_4 - k_1l_3) \\ (k_2l_4 - k_1l_3) & (k_1l_3^2 + k_2l_4^2) \end{pmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

When $k_2l_4 - k_1l_3 = 0$ or $k_2l_4 = k_1l_3$ then the system is statically decoupled but dynamically coupled in which the equation of motion will be which was

$$\begin{pmatrix} M & me \\ me & J \end{pmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{pmatrix} (k_1 + k_2) & 0 \\ 0 & (k_1 l_3^2 + k_2 l_4^2) \end{pmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Static/Stiffness and Dynamic/Mass Coupling: If there is a point 'C' in the system along which a displacement produces pure translation along the line of action of spring force as shown in figure-6.28

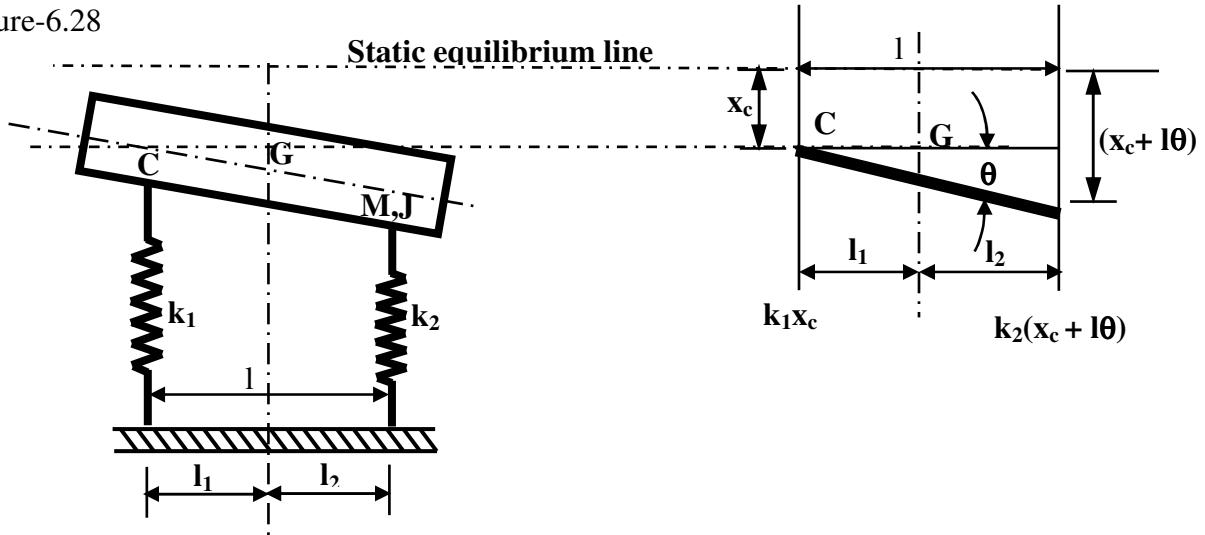


Figure-6.28 (a) system under vibration

(b) displacements at the springs

Then the equation of motion is

$$M\ddot{x}_c + ml_1\ddot{\theta} = -k_1x_c - k_2(x_c + l\theta) \quad \text{and} \quad J\ddot{\theta} + ml_1\ddot{x}_c = -k_2(x_c + l\theta)l$$

Rearranging the above two equation we have

$$M\ddot{x}_c + ml_1\ddot{\theta} + (k_1 + k_2)x_c + k_2l = 0$$

$$J\ddot{\theta} + ml_1\ddot{x}_c + k_2lx_c + k_2l^2\theta = 0$$

The above equation can be written in matrix form as

$$\begin{pmatrix} M & ml_1 \\ ml_1 & J \end{pmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{pmatrix} (k_1 + k_2) & k_2l \\ k_2l & k_2l^2 \end{pmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In which both the mass matrix and stiffness matrix are non-diagonal matrix hence the system is both statically and dynamically coupled.

Determine the normal mode of vibration of an automobile shown in figure-6.29 simulated by a simplified two degree of freedom system with the following numerical values $m = 1460 \text{ kg}$, $L_1 = 1.35\text{m}$, $L_2 = 2.65 \text{ m}$, $K_1 = 4.2 \times 10^5 \text{ N/m}$, $K_2 = 4.55 \times 10^5 \text{ N/m}$ and $J = mr^2$ where $r = 1.22 \text{ m}$

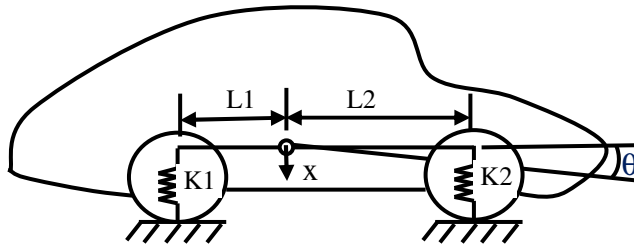


Figure-6.29

Automobile can be modeled as shown in figure -6.30

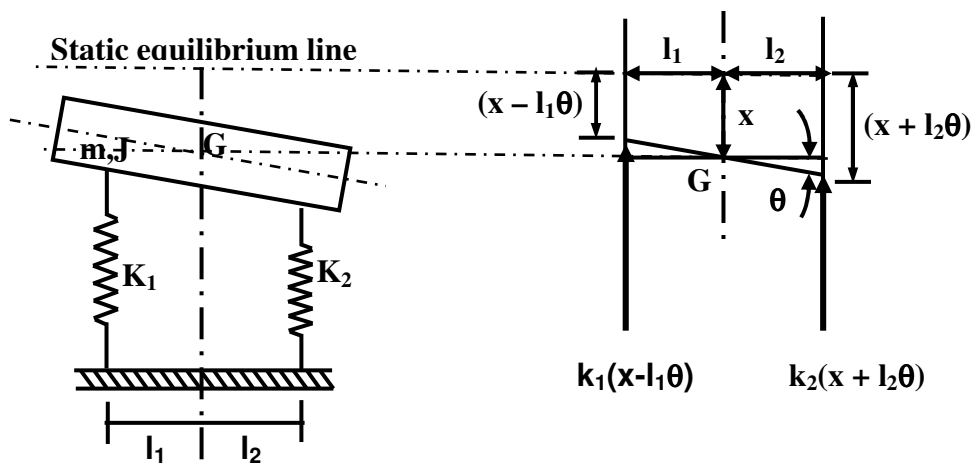


Figure-6.30 (a) system under vibration (b) displacements at the springs

Let at any given instant of time the translatory displacement be 'x' and an angular displacement be 'theta' from its equilibrium position of the automobile. Then the left spring with the stiffness k_1 and the right spring with the stiffness k_2 are compressed through $(x - l_1\theta)$ and $(x + l_2\theta)$ from their equilibrium position, The forces acting on the system are as shown in the free body diagram in figure-6.30(b). The differential equation of motion of the automobile in 'x' and 'theta' direction are written by considering the forces and moments in their respective direction.

Thus we have the equation of motion

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \text{ ----- (1)}$$

$$J\ddot{\theta} = +k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2 \text{ ----- (2)}$$

Rearranging the above two equation we have

$$m\ddot{x} + (k_1 + k_2)x = (k_1l_1 - k_2l_2)\theta \text{ ----- (3)}$$

$$J\ddot{\theta} + (k_1l_1^2 + k_2l_2^2)\theta = (k_1l_1 - k_2l_2)x \text{ ----- (4)}$$

Equation (3) and (4) are the second order differential equation and the solution for x and θ are obtained by considering that they can have harmonic vibration under steady state condition. Then considering the case when the system execute harmonic vibration at frequency ω

Thus if $x = A\sin\omega t$, and $\theta = B\sin\omega t$ ---- (5) Substitute equation (5) into the equation (3) and (4) and canceling the common term $\sin\omega t$ we get

$$[-m\omega^2 + (k_1 + k_2)]A = (k_1l_1 - k_2l_2)B \text{ ----- (6)}$$

$$[-J\omega^2 + (k_1l_1^2 + k_2l_2^2)]B = (k_1l_1 - k_2l_2)A \text{ ---- (7)}$$

From equation (6) we have $A/B = (k_1l_1 - k_2l_2) / [(k_1 + k_2) - m\omega^2]$ ----- (8)

From equation (7) we have $A/B = [(k_1l_1^2 + k_2l_2^2) - J\omega^2] / (k_1l_1 - k_2l_2)$ ---- (9)

Equating (8) and (9) $A/B = (k_1l_1 - k_2l_2) / [(k_1 + k_2) - m\omega^2] = [(k_1l_1^2 + k_2l_2^2) - J\omega^2] / (k_1l_1 - k_2l_2)$

$[(k_1 + k_2) - m\omega^2][(k_1l_1^2 + k_2l_2^2) - J\omega^2] = (k_1l_1 - k_2l_2)^2$ Further Simplification will give

$$mJ\omega^4 - [J(k_1 + k_2) + m(k_1l_1^2 + k_2l_2^2)]\omega^2 + k_1k_2(l_1 + l_2)^2 = 0 \text{ ----- (10)}$$

Substituting the value of m, J, k₁, k₂, l₁, l₂ into the above equation (10) we have

$$3.173 \times 10^6 \omega^4 - 4.831 \times 10^9 \omega^2 + 1.72 \times 10^{12} = 0 \text{ --- (11) or } \omega^4 - 1.523 \times 10^3 \omega^2 + 5.429 \times 10^5 = 0 \text{ ---- (12)}$$

Letting $\omega^2 = \lambda$ we have $\lambda^2 - 1.523 \times 10^3 \lambda + 5.429 \times 10^5 = 0$ ---- (13)

Equation (13) is quadratic equation in λ. Thus solving equation (13) we get two roots which are

$\lambda_1 = 569.59$, $\lambda_2 = 953.13$ Since $\omega^2 = \lambda$ we have $\omega = \sqrt{\lambda}$. Thus $\omega_1 = 23.86$ rad/sec and $\omega_2 = 30.87$ rad/sec

Thus $f_{n1} = 3.797$ Hz and $f_{n2} = 4.911$ Hz

Un-damped Dynamic Vibration Absorber

Consider a two degree of freedom system with a forcing function $F_1 = F_0 \sin\omega t$ as shown in figure-6.31(a).

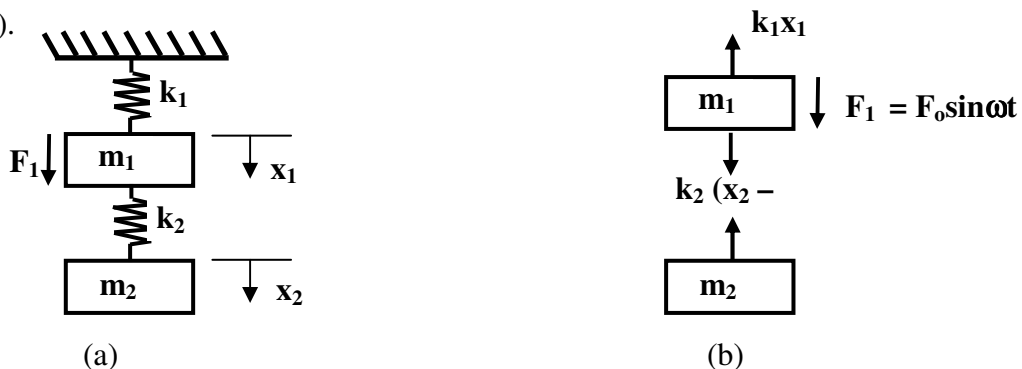


Figure- 6.31(a) Two degree of freedom system with forcing function F₁ on mass 1

Let x_1 and x_2 be the displacement of the masses m_1 and m_2 respectively at any given instant of time measured from the equilibrium position with $x_2 > x_1$. Then the spring forces acting on the masses are as shown in free body diagram in Figure 6.31(b)

Based on Newton's second law of motion $\sum f = m \ddot{x}$

For mass m_1 we have

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + F_0 \sin \omega t$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 x_2 + k_2 x_1 = F_0 \sin \omega t \quad \text{----- (1)}$$

for mass (2)

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$m_2 \ddot{x}_2 + k_2 x_2 = k_2 x_1 \quad \text{----- (2)}$$

The solution for x_1 and x_2 are obtained by considering that the masses execute harmonic vibration at frequency ω . Thus if $x_1 = X_1 \sin \omega t$, and $x_2 = X_2 \sin \omega t$ ----- (3)

Then we have $\ddot{x}_1 = -\omega^2 X_1 \sin \omega t$ and $\ddot{x}_2 = -\omega^2 X_2 \sin \omega t$ ----- (4)

Substituting equation (3) and (4) into the equation (1) and (2) we get

$$-m_1 \omega^2 X_1 \sin \omega t + (k_1 + k_2) X_1 \sin \omega t = k_2 X_2 \sin \omega t + F_0 \sin \omega t \quad \text{----- (5)}$$

$$-m_2 \omega^2 X_2 \sin \omega t + k_2 X_2 \sin \omega t = k_2 X_1 \sin \omega t \quad \text{----- (6)}$$

Canceling the common term $\sin \omega t$ on both the sides of equation (5) and (6) we have

$$[(k_1 + k_2) - m_1 \omega^2] X_1 - k_2 X_2 = F_0 \quad \text{----- (7)}$$

$$k_2 X_1 - [k_2 - m_2 \omega^2] X_2 = 0 \quad \text{----- (8)}$$

Solving for X_1 and X_2 by cramer's rule

$$X_1 = \frac{\begin{vmatrix} F_0 & -K_2 \\ 0 & K_2 - m_2 \omega^2 \end{vmatrix}}{\Delta} \quad \text{----- (9)}$$

$$X_2 = \frac{\begin{vmatrix} (K_1 + K_2) - m_1 \omega^2 & F_0 \\ -K_2 & 0 \end{vmatrix}}{\Delta} \quad \text{----- (10)}$$

where Δ is the determinant of characteristic equations.

$$\Delta = \begin{vmatrix} (K_1 + K_2) - m_1 \omega^2 & -K_2 \\ -K_2 & K_2 - m_2 \omega^2 \end{vmatrix} \neq 0 \quad \text{----- (11)}$$

Solving the above determinant we get $\Delta = \{(K_1 + K_2) - m_1 \omega^2\} \{K_2 - m_2 \omega^2\} - K_2^2$ ----- (12)

If the two vibratory masses are considered separately as shown in Figure- 6.32, the mass 1 is a main system and mass 2 is an secondary system. This system can be used as Dynamic vibration absorber or Tuned damper by using the amplitude Equations (9) and (10).

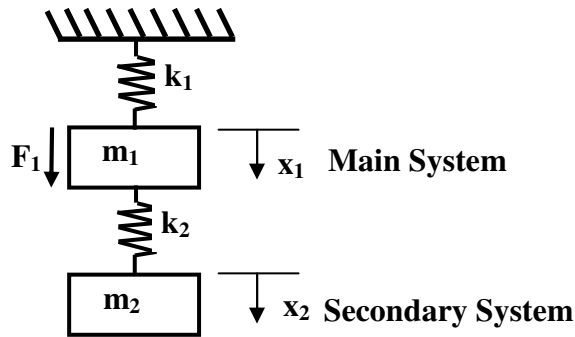


Figure- 6.32

If the system has to be used as a Dynamic vibration absorber, then the amplitude of vibration of mass m_1 should be equal to zero, i.e $X_1=0$.

$$X_1 = \frac{\begin{vmatrix} F_0 & -K_2 \\ 0 & K_2 - m_2\omega^2 \end{vmatrix}}{\Delta} = 0 \quad \text{----- (13) Then we have} \quad \begin{vmatrix} F_0 & -K_2 \\ 0 & K_2 - m_2\omega^2 \end{vmatrix} = 0$$

$$F_0(k_2 - m_2\omega^2) = 0 \quad \text{since } F_0 \text{ cannot be equal to zero we have} \quad \mathbf{k_2 - m_2\omega^2 = 0}$$

$$\omega^2 = k_2/m_2 \quad \text{or} \quad \omega = \sqrt{k_2/m_2} \text{ rad/sec} \quad \text{----- (14)}$$

The above Eqn. is the natural frequency of secondary or absorber system.