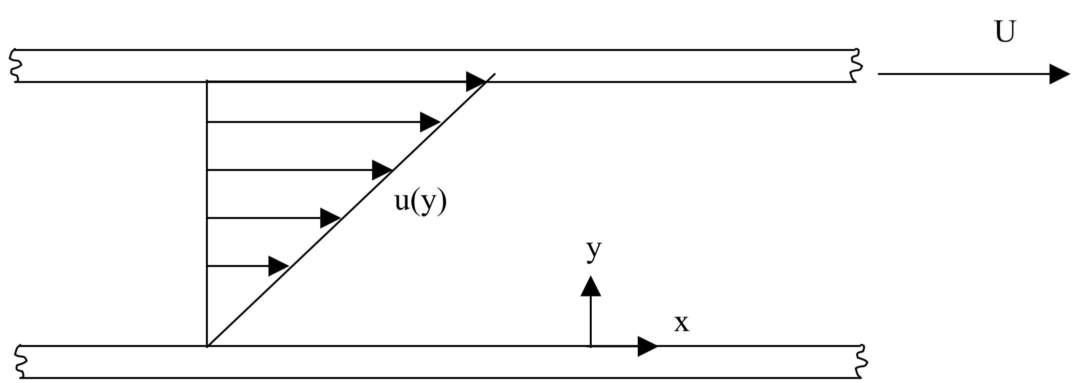
Viscous Flow

Up to this point in your fluid dynamics education, from Basic Fluid Mechanics, to Fundamental Aerodynamics, to Gas Dynamics to this class, you have studied inviscid flows (with the exception of the first days of BFM). Unfortunately, there are no inviscid flows, all flows are viscous and most practical flows are turbulent (unsteady). In this section we turn our attention to viscous flows and begin the study of boundary layer theory.

We'll begin by reviewing what you learned at the beginning of Basic Fluid Mechanics, move quickly through the basic definitions and variables associated with viscous flows and boundary layers, and then discuss an airfoil analysis tool that incorporates viscous effects. The goals of this effort is to complete the drag picture via computation of the skin friction. To do this let us return to the first few days of Basic Fluid Mechanics.

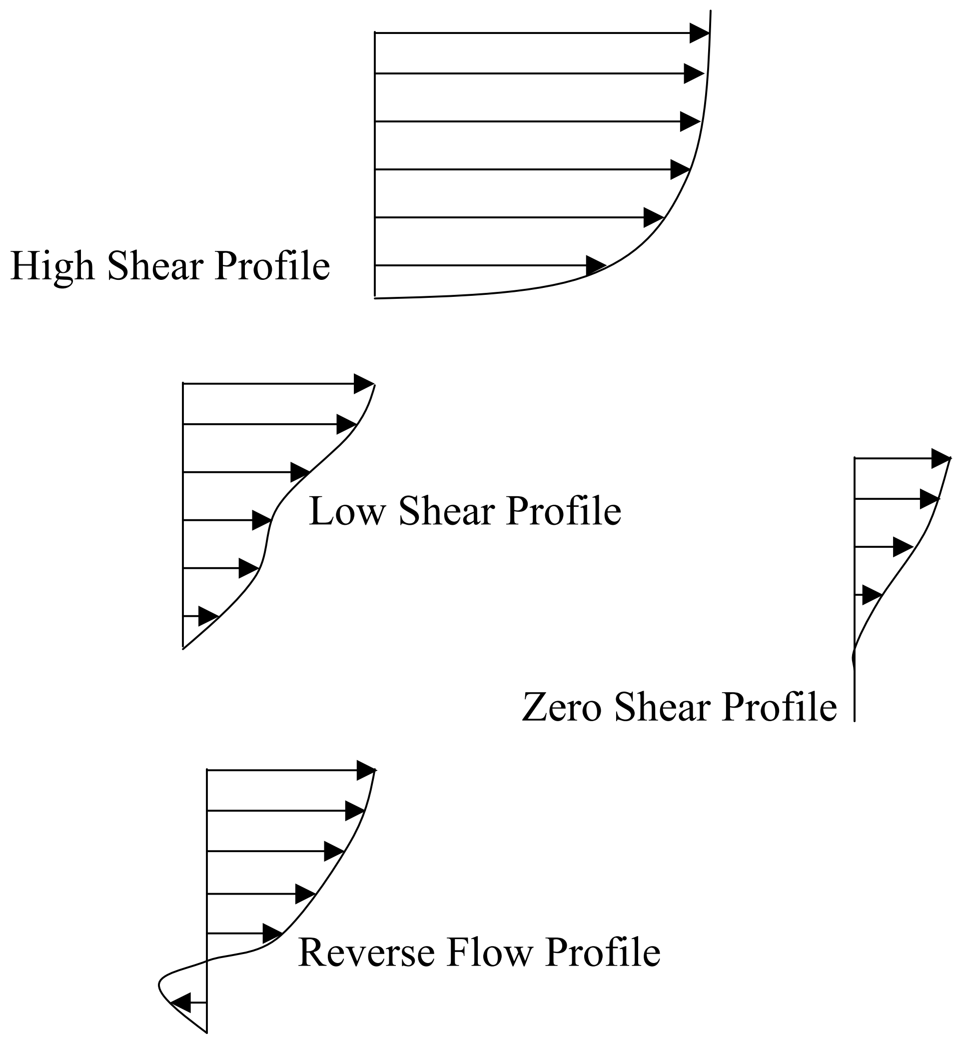
Consider the flow between two infinite parallel plates; the bottom is stationary while the top moves to the right with speed *U*.



We were told that viscosity causes the fluid to stick to the plate, hence its velocity is equal to that of the plate. If the flow has zero pressure gradient, i.e., $\frac{dp}{dx} = 0$, then a linear velocity profile emerges. This is called Couette flow. One is usually asked to calculate the viscous resistance of the fluid on the walls or vice versa given the knowledge that

$$\tau = \mu \frac{du}{dy} \tag{15.1}$$

where τ is the shear stress, which has units of pressure, force/area, and μ is the molecular viscosity, a function of the working fluid and temperature. The velocity gradient is the inverse of the slope of the line drawn in the figure.



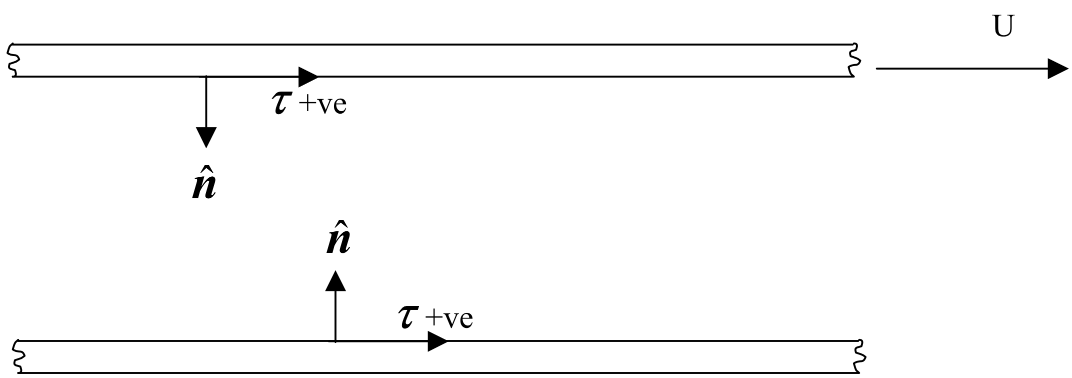
Facts about Couette flow:

1. The shear stress always acts in the direction opposite to the flow.
2. The bottom wall sees fluid moving in the positive x-direction.
3. The top wall sees fluid moving in the negative x-direction.

This is at first confusing, but some order is brought about by the right hand rule.

If you use your right hand and allow the thumb to point in the direction of the surface normal, the index finger will point in the direction of positive shear stress.

Consider the plates



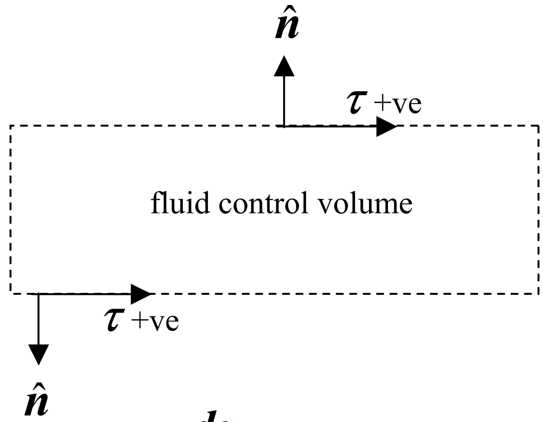
Is this consistent with $\frac{du}{dy}$'s sign?

Yes, all along the profile u grows as y increases, so $\frac{du}{dy} > 0$.

Is this consistent with the effect of the fluid on the walls?

Yes, the fluid tends to pull the bottom wall to the right but slows down or retards the motion of the top wall.

What about from the perspective of the fluid, does the convention still make sense?

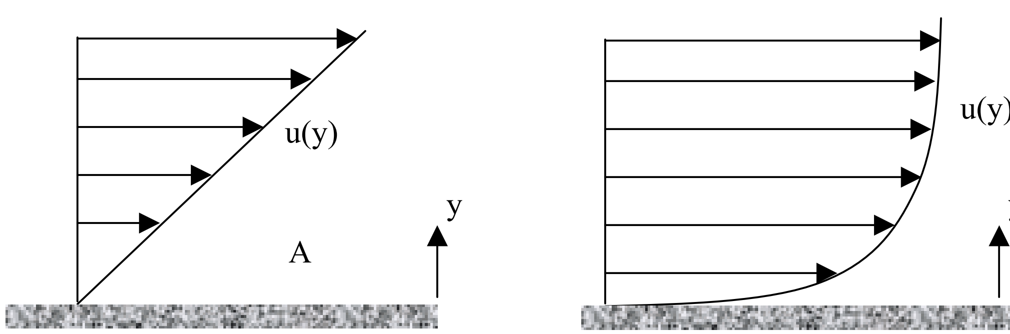


Is this consistent with $\dfrac{du}{dy}$'s sign and the direction of the forces on the fluid?

Yes, the top wall imparts onto the fluid a force in the $+x$ direction while it imparts a force in the $-x$ direction on the bottom wall.

Common Error:

We are used to thinking of slope in terms of $\dfrac{dy}{dx}$, but we have $\dfrac{du}{dy}$ and tend to illustrate it physically, that is with u in the $+x$-direction and y in its natural direction. Hence,

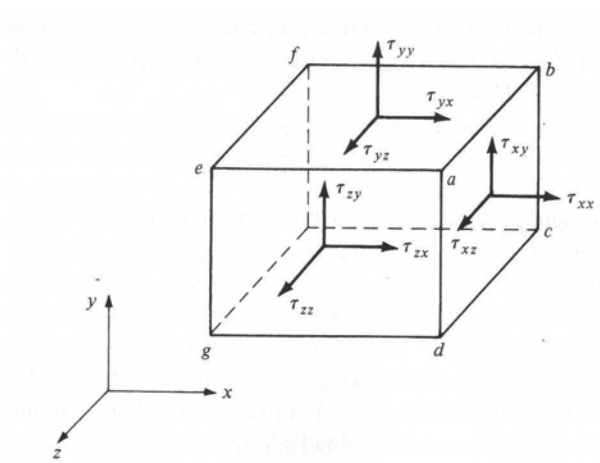


$\tau_A < \tau_B$ at the wall

To keep it straight just remember the discrete approximation $\frac{du}{dy} \approx \frac{\Delta u}{\Delta y}$ and ask yourself the question, "how much does *u* change as *y* increases?"

The Couette flow discussed during the first classes of Basic Fluid Mechanics provides some good insight into the effects of viscosity, however, it is way over simplified in its presentation.

1. τ is not just a property that exists at the walls, it exists everywhere in the fluid. $\tau = \tau(y)$ for this problem.
2. τ is not a scalar or a vector, it is actually a tensor, that is, it has 9 components and is defined only once a plane surface is provided in the flow.



In its tensor form τ is associated with 2 direction and given 2 subscripts. The first subscript refers to the normal direction of the plane defining it and the second refers to the direction in which the force/area acts. In the Couette flow example, $\tau = \tau_{yx}$, a force defines by a plane whose normal is in the *y*-direction and which acts in the *x*-direction.

Note that

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{15.2}$$

we had $v = 0$ in the Couette flow so τ simplified considerably. In the general case

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{15.3}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Normal stresses are different from but in the same direction as the pressure.

$$\tau_{xx} = \lambda\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{\nabla \cdot \vec{V}} + 2\mu\left(\frac{\partial u}{\partial x}\right) \tag{15.4}$$

$$\tau_{yy} = \lambda\left(\nabla \cdot \vec{V}\right) + 2\mu\left(\frac{\partial v}{\partial y}\right) \tag{15.5}$$

$$\tau_{zz} = \lambda\left(\nabla \cdot \vec{V}\right) + 2\mu\left(\frac{\partial w}{\partial z}\right) \tag{15.6}$$

λ is the *bulk viscosity* which by assumption is

$$\lambda = -\frac{2}{3}\mu \tag{15.7}$$

via Stokes hypothesis.

Newton's second law can now be applied to a viscous fluid element by considering the shear stresses in addition to pressure to derive the governing, *Navier-Stokes* equation.

Skin Friction Coefficient

As with lift and drag, it is useful to nondimensionalize the shear stress. We recall that τ has units of pressure, i.e., force/area. However, unlike the pressure coefficients, which is nondimensionalized by the dynamic pressure of the freestream, q_∞, the edge properties are the more important parameters. Therefore, the skin friction coefficient is the shear stress normalized by the boundary layer edge dynamic pressure:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} \tag{16.1}$$

Clearly C_f is a function of the location along the surface.

Reynolds Number

Using the idea of edge properties it is possible to define a Reynolds number:

$$R_{eL} = \frac{\rho_e u_e L}{\mu_e} \tag{16.2}$$

where L is the length along the surface. This idea can be extended and used as a way of nondimensionalizing the axial coordinate:

$$R_{ex} = \frac{\rho_e u_e x}{\mu_e} \tag{16.3}$$

For a linear velocity profile like that in Couette flow with plates separated by a distance *D*, we see that

$$\tau_w = \mu_e \frac{u_e}{D} \tag{16.4}$$

The $R_{e\,x}$ and $R_{e\,L}$ definitions are useless in this case, since the plates are of infinite extent. Instead we introduce

$$R_{e\,D} = \frac{\rho_e u_e D}{\mu_e} \tag{16.5}$$

Combining Eqs. (16.4) and (16.5) for Couette flow, we see

$$C_f = \frac{\mu\left(u_e / D\right)}{\frac{1}{2}\rho_e u_e^2} = \frac{2\mu}{\rho_e u_e D}$$
$$C_f = 2 \big/ R_{eD} \tag{16.6}$$

Which is, of course, true only for Couette flow, but demonstrates a basic idea that

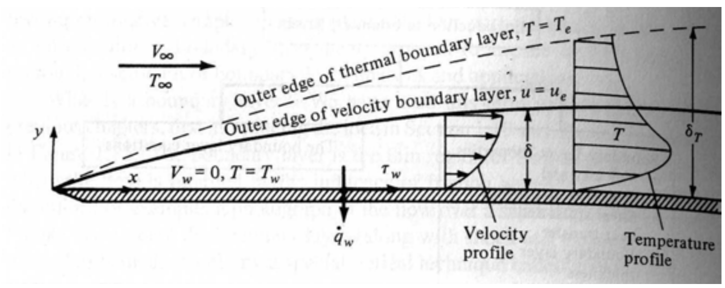
$$C_f \propto \frac{1}{R_e} \tag{16.7}$$

showing that as Reynolds number goes up, the skin friction goes down.

Boundary Layer Concept

The Couette flow example is one of a fully viscous flow. In that case, the shear stress is constant throughout the region. This is not true in the general case of viscous flows, particularly external flows, and leads to the viscous flow concept: *the boundary layer.*

Boundary Layer – The region of the flow close to a wall in which viscous effects are dominant.



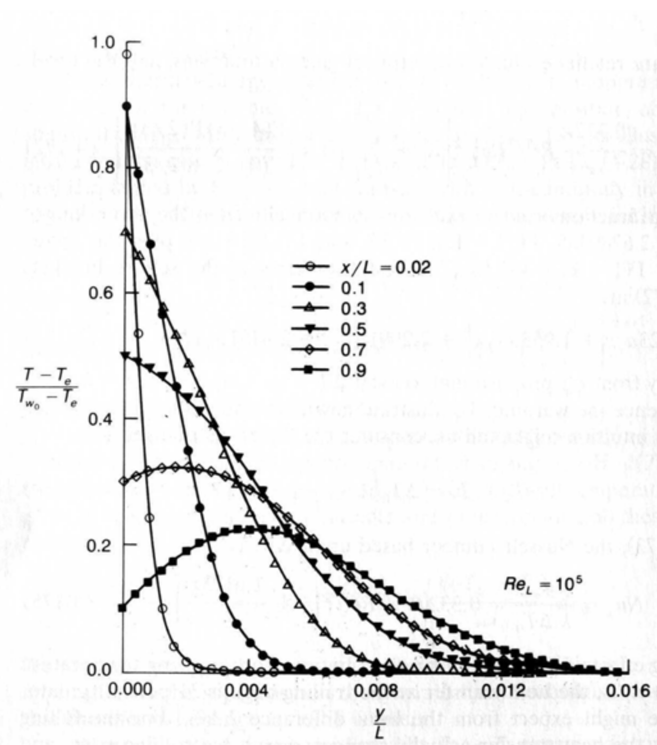
The boundary layer essentially divides regions of the flow dominated by viscous effects from those dominated by inviscid effects. Unfortunately, the definition of a boundary layer is a little fuzzy, in that its extent depends upon the effects one is pursuing, i.e., velocity, mass flow, momentum or temperature. To that end, several boundary layer thickness definitions exist:

Boundary layer thickness - δ_{99}. The normal direction distance from the surface at which the local velocity achieves 99% of the edge value.

This definition is clearly arbitrary, there exists a δ_{90}, a δ_{95}, etc., but is useful in defining the extent of the so-called *velocity boundary layer.*

Thermal boundary layer thickness -δ_T. The distance from the surface at which the local temperature achieves 99% of the edge temperature.

This definition is even harder to quantify because the wall boundary condition is more flexible. That is the temperature can be higher or lower than the edge conditions, or the wall can be insulated, i.e., adiabatic. In general, the temperature increases in the velocity boundary layer because viscous effects convert the energy in the flow to heat. However, wall heat transfer can still suck that energy out, allowing for a wide variety of possible profiles.



The arbitrary nature of the velocity and thermal boundary layer definitions left researchers searching for more quantitative measures. A general observation of viscous flows based on conservation of mass ideas is that the velocity "deficit" caused by the viscous walls forces incoming mass to exit above the plate, i.e., inducing a v-velocity component. This, in effect, creates a *streamline displacement* effect that moves the external flow streamlines a finite distance. This distance is not the same as δ_{99}.

Displacement thickness - δ^* - The normal direction distance from the surface that an otherwise undisturbed streamline would be displaced because of the effect of viscous walls.

The basic idea comes from control volume theory and utilizes velocity deficit ideas.

If

$$A = \int_0^{y_1} \rho u dy$$ - mass flow between ***0*** and y_1.

$$B = \int_0^{y_1} \rho_e u_e dy$$ - mass flow between ***0*** and y_1 if no viscous effects are present

$$B - A = \int_0^{y_1} (\rho_e u_e - \rho u) dy$$ - mass flow deficit (17.1)

This mass flow deficit is equated to an inviscid mass flow across a new distance from the wall called the *displacement thickness.*

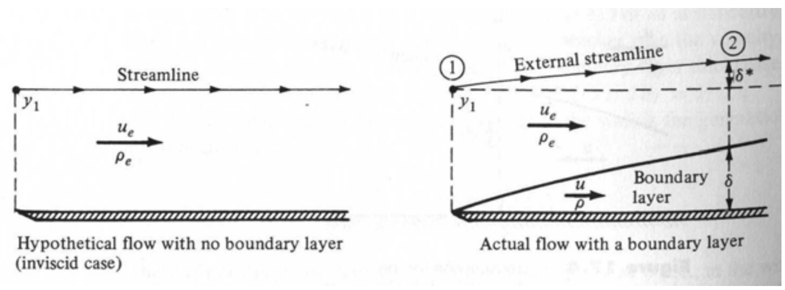
$$B - A = \rho_e u_e \delta^* \qquad (17.2)$$

Equating Eqs. (17.1) and (17.2) gives

$$\rho_e u_e \delta^* = \int_0^{y_1} (\rho_e u_e - \rho u) dy$$

$$\delta^* \equiv \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \qquad (17.3)$$

It's physical interpretation is given in the figures below



Unfortunately, the story does not end here, since the deficit associated with mass flow is not the same as the deficit associated with momentum flow. A similar thickness is defined for momentum by using the idea of the difference in momentum carried by the edge velocity.

$A = \int_0^{y_1} \rho u^2 dy$ - momentum flow between 0 and y_1 carried by the actual velocity..

$B = \int_0^{y_1} \rho u u_e dy$ - momentum flow between 0 and y_1 carried by a fictitious edge velocity.

$$B - A = \int_0^{y_1} \rho u (u_e - u) dy$$ - momentum flow deficit

$$B - A = \rho_e u_e^2 \theta$$ - inviscid momentum flow

$$\theta \equiv \int_0^{y_1} \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy \qquad (17.4)$$

Momentum Thickness

The *momentum thickness,* θ, is an important parameter for drag prediction and skin friction because it represents a momentum deficit, i.e., drag. It is proportional to the skin friction.

Reynolds Numbers

The three new distances lead to three new Reynolds numbers

$$R_{e\delta} \equiv \frac{\rho_e u_e \delta}{\mu_e}$$ Boundary layer thickness Reynolds number

$$R_{e\delta^*} \equiv \frac{\rho_e u_e \delta^*}{\mu_e}$$ Displacement thickness Reynolds number

$$R_{e\theta} \equiv \frac{\rho_e u_e \theta}{\mu_e}$$ Momentum thickness Reynolds number

These numbers are particularly useful for correlating boundary layer parameters and the transition to turbulence.

<u>Shape Factor</u>

Another important parameter is the *shape factor*.

$$\boldsymbol{H} \equiv \frac{\delta^*}{\theta}$$

It is generally true that $\delta > \delta^* > \theta$.

Derivation of the Boundary Layer Equations

The governing equations of viscous fluid mechanics are the Navier-Stokes equations. A greatly simplified set of governing equations can be derived in the boundary layer region of a viscous flow. This descends from knowledge of the flow field variables in this region. Start with the Navier-Stokes equations, written in nondimensional form for the x-momentum equation:

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] \quad (17.5)$$

This form of the governing equations comes about by utilizing nondimensionalized variables

$$\rho' = \frac{\rho}{\rho_\infty}, u' = \frac{u}{V_\infty}, v' = \frac{v}{V_\infty}, p' = \frac{p}{p_\infty}, \mu' = \frac{\mu}{\mu_\infty}, x' = \frac{x}{c}, y' = \frac{y}{c}$$

Start with the dimensional form

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \quad (17.6)$$

Substitution of the nondimensional variables gives

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\left(\frac{p_\infty}{\rho_\infty V_\infty^2}\right)\frac{\partial p'}{\partial x'} + \left(\frac{\mu_\infty}{\rho_\infty V_\infty c}\right)\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] \quad (17.7)$$

where

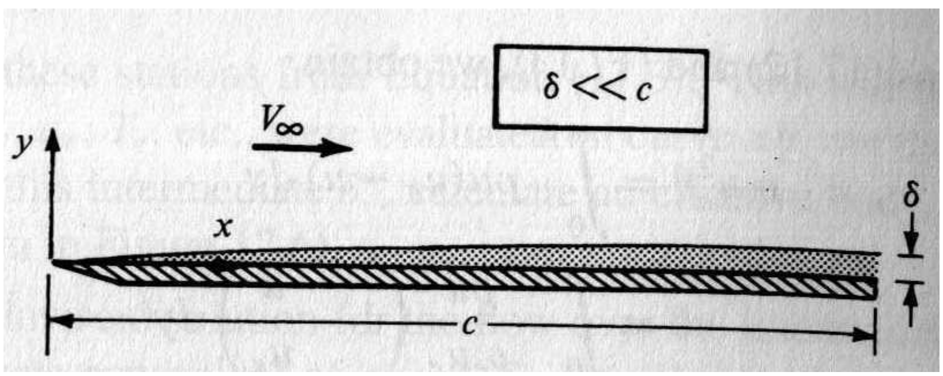
$$\frac{p_\infty}{\rho_\infty V_\infty^2} = \frac{\gamma p_\infty}{\gamma \rho_\infty V_\infty^2} = \frac{a_\infty^2}{\gamma V_\infty^2} = \frac{1}{\gamma M_\infty^2}$$

and

$$\frac{\mu_\infty}{\rho_\infty V_\infty c} = \frac{1}{\mathrm{Re}_\infty}$$

The idea of using nondimensional variables is to compare the size of specific terms in the equations. In this way, one can determine which terms to eliminate.

Fundamental assumption: The Boundary Layer is very thin compared to the body.



If c is the chord length then $\delta << c$.

Variable	Variation in BL		
u'	$0 \rightarrow 1$	$\Rightarrow$	$u' = O(1)$

$\rho' = O(1)$

$x' = O(1)$

$y' = O\left(\delta/c\right) = O(\delta)$

The continuity equation then says

$$\frac{\partial \rho' u'}{\partial x'} + \frac{\partial \rho' v'}{\partial y'} = 0 \qquad (17.8)$$

$$\underbrace{\frac{O(1)O(1)}{O(1)}}_{A} + \underbrace{\frac{O(1)v'}{O(\delta)}}_{B} = 0$$

Therefore for term A to be about the same size as term B, we must have

$$v' = O(\delta) \qquad (17.9)$$

Hence terms in Equation (17.5) become

$$O(1)O(1)\frac{O(1)}{O(1)} + O(1)O(\delta)\frac{O(1)}{O(\delta)} = -\frac{1}{\gamma M_\infty^2}\frac{O(1)}{O(1)} + \frac{1}{\mathrm{Re}_\infty}\left[\frac{1}{O(\delta)}\left[O(1)\left(\frac{O(\delta)}{O(1)} + \frac{O(1)}{O(\delta)}\right)\right]\right]$$

We then assume the Re is very large such that $\frac{1}{\mathrm{Re}_\infty} = O(\delta^2)$, so that

$$O(1) + O(1) = O(1) + O(\delta^2)\left[\frac{1}{O(\delta)}\left[O(1)\left(O(\delta) + \frac{O(1)}{O(\delta)}\right)\right]\right]$$

$$O(1) + O(1) = O(1) + O(\delta^2) + O(1)$$

There is then one term in the equation that is much smaller than the others and can be eliminated, thereby simplifying the equations greatly.

Upon simplification the equations become

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left[\mu' \frac{\partial u'}{\partial y'} \right] \quad (17.10)$$

A similar analysis can be performed for the y-momentum equation.

$$\rho' u' \frac{\partial v'}{\partial x'} + \rho' v' \frac{\partial v'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial y'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial x'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

$$O(1)O(1)\frac{O(\delta)}{O(1)} + O(1)O(\delta)\frac{O(\delta)}{O(\delta)} = \frac{O(1)}{O(\delta)} + O(\delta^2)\frac{1}{O(1)}\left[O(1)\left[\frac{O(\delta)}{O(1)} + \frac{O(1)}{O(\delta)} \right] \right]$$

Taken together we get

$$O(\delta) + O(\delta) = O\left(\frac{1}{\delta} \right) + O(\delta^3) + O(\delta)$$

Where there is only one **dominant term**. Which implies to first approximation that that term is the only one that remains.

$$\frac{\partial p}{\partial y} = 0 \quad (17.11)$$

Which is the y-momentum equation for a boundary layer.

A similar analysis can be done for the energy equation resulting in the complete set of boundary layer equations:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial x} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial y} = 0 \qquad (17.12)$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial x} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + u \frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2$$

with boundary conditions

wall:	$y=0$	$u=0$	$v=0$	$T=T_w$
edge:	$y\to\infty$	$u\to u_e$	$T\to T_e$	

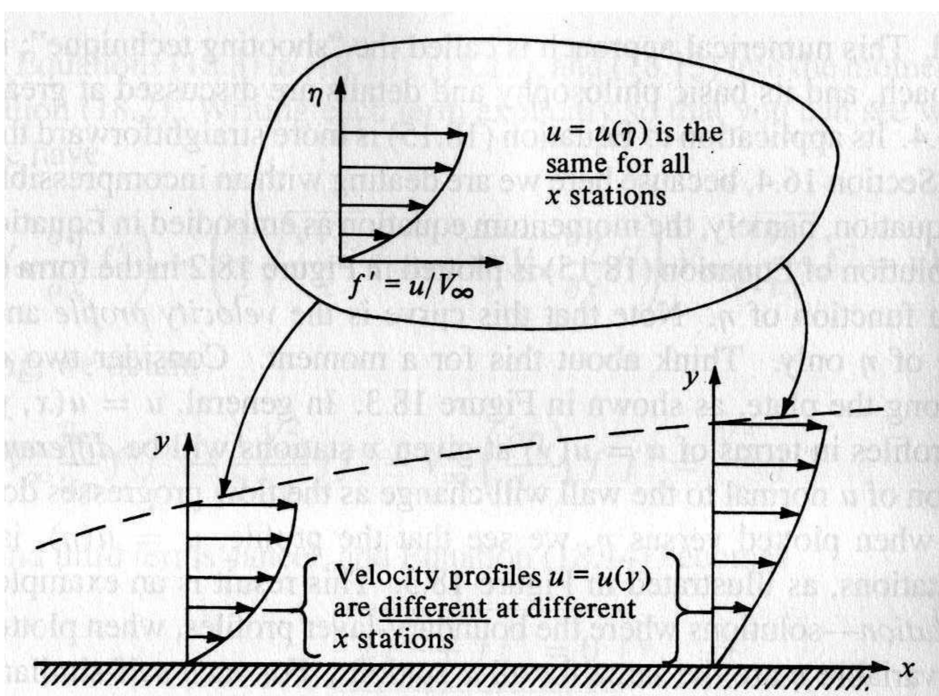
Similarity Solutions

The development of the boundary layer equations simplifies considerably the governing equations and in some cases makes them tractable to analytical solution. One such approach to defining solutions is to employ a transformation of variables so that in some sense every profile is similar to another. This type of solution is called a *similarity solution*. One place where this leads to useful solutions is the laminar flat plate boundary layer.

Blasius Laminar Flat Plate Solution

The governing equations are transformed by the introduction of the *similarity parameters* η and ξ, where

$$\xi = x, \qquad \eta = y\sqrt{\frac{V_\infty}{\upsilon x}}, \qquad f'(\eta) = \frac{u}{V_\infty}$$



Under this transformation the boundary layer equations reduce to:

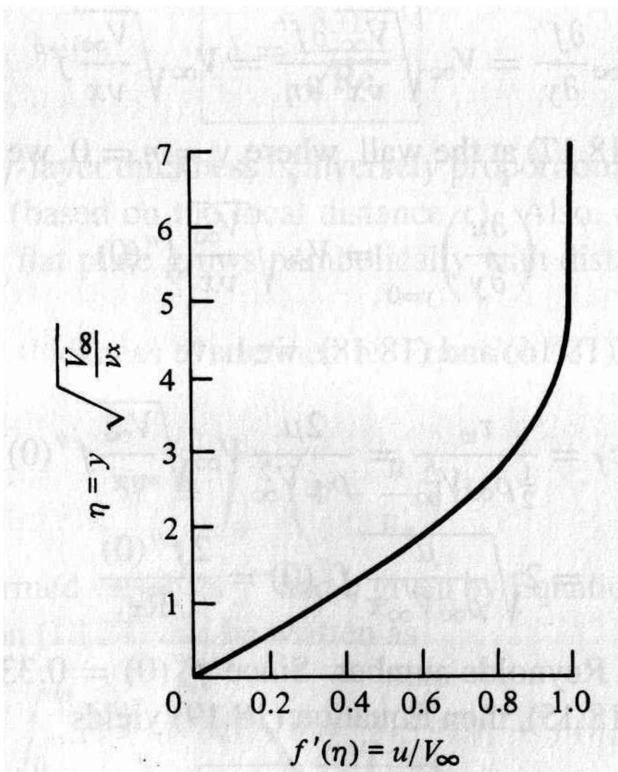
$$2f''' + ff'' = 0 \qquad (18.1)$$

A much simpler ODE.

With BCs $\eta=0$: $f=0$ $f'=0$

$\eta\to\infty$: $f'=1$

The Blasius boundary layer profile for a laminar flat plate in these coordinates is then



The solution to Equation (18.1) is done in a numerical fashion and values of f, f' and f'' are provided in tabulated form (next page). The text demonstrates how the profiles of f and its derivatives can be used to determine properties of the boundary layer such as the skin friction coefficient and the boundary layer thickness. It is suggested that you review that material so that you will be able to calculate these parameters.

TABLE 4-1
Numerical solution of the Blasius flat-plate relation, Eq. (4-45)

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0.0	0.0	0.46960
0.1	0.00235	0.04696	0.46956
0.2	0.00939	0.09391	0.46931
0.3	0.02113	0.14081	0.46861
0.4	0.03755	0.18761	0.46725
0.5	0.05864	0.23423	0.46503
0.6	0.08439	0.28058	0.46173
0.7	0.11474	0.32653	0.45718
0.8	0.14967	0.37196	0.45119
0.9	0.18911	0.41672	0.44363
1.0	0.23299	0.46063	0.43438
1.1	0.28121	0.50354	0.42337
1.2	0.33366	0.54525	0.41057
1.3	0.39021	0.58559	0.39598
1.4	0.45072	0.62439	0.37969
1.5	0.51503	0.66147	0.36180
1.6	0.58296	0.69670	0.34249
1.7	0.65430	0.72993	0.32195
1.8	0.72887	0.76106	0.30045
1.9	0.80644	0.79000	0.27825
2.0	0.88680	0.81669	0.25567
2.2	1.05495	0.86330	0.21058
2.4	1.23153	0.90107	0.16756
2.6	1.41482	0.93060	0.12861
2.8	1.60328	0.95288	0.09511
3.0	1.79557	0.96905	0.06771
3.2	1.99058	0.98037	0.04637
3.4	2.18747	0.98797	0.03054
3.6	2.38559	0.99289	0.01933
3.8	2.58450	0.99594	0.01176
4.0	2.78388	0.99777	0.00687
4.2	2.98355	0.99882	0.00386
4.4	3.18338	0.99940	0.00208
4.6	3.38329	0.99970	0.00108
4.8	3.58325	0.99986	0.00054
5.0	3.78323	0.99994	0.00026
5.2	3.98322	0.999971	0.000119
5.4	4.18322	0.999988	0.000052
5.6	4.38322	0.999995	0.000022
5.8	4.58322	0.999998	0.000009
6.0	4.78322	0.999999	0.000003

From this tabulated data the following can be determined:

$$C_f = \frac{0.664}{\sqrt{\text{Re}_x}}$$ Local Skin Friction Coefficient

$$C_f = \frac{1.328}{\sqrt{\text{Re}_c}}$$ Total Skin Friction Coefficient for a flat plate of length C

$$\delta = \frac{5.0x}{\sqrt{\text{Re}_x}}$$ Boundary Layer Thickness

$$\delta^* = \frac{1.72x}{\sqrt{\text{Re}_x}}$$ Displacement Thickness

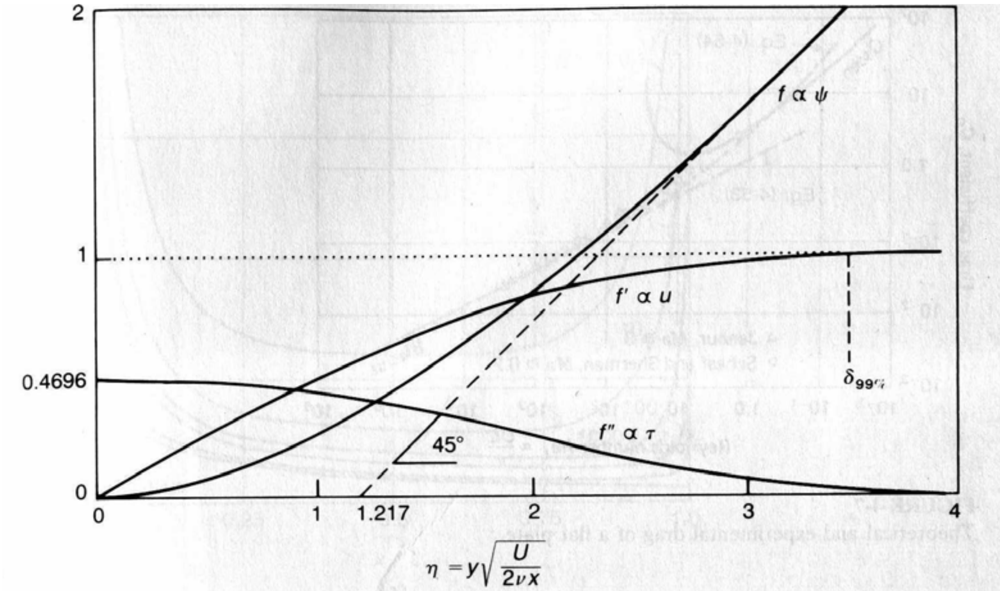
$$\theta = \frac{0.664x}{\sqrt{\text{Re}_x}}$$ Momentum Thickness

Note that the total skin friction for the flat plate is related to the momentum thickness via

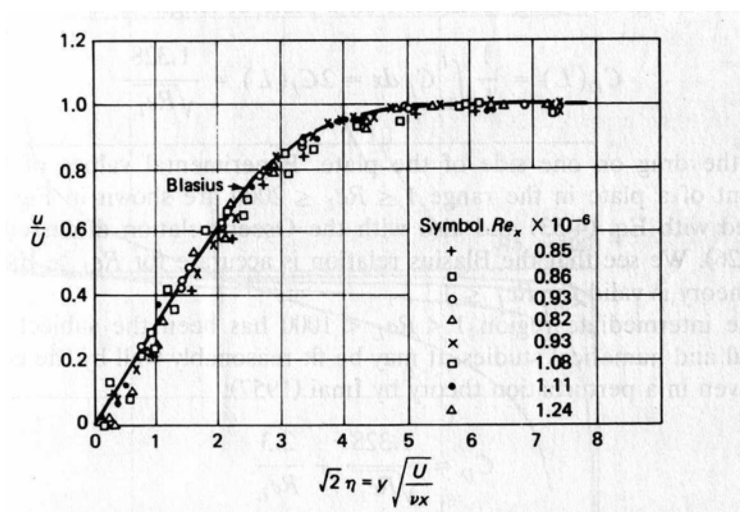
$$C_f = \frac{2\theta_{x=c}}{c}$$

Indicating that the skin friction drag coefficient is directly proportional to the value of the momentum thickness at the trailing edge.

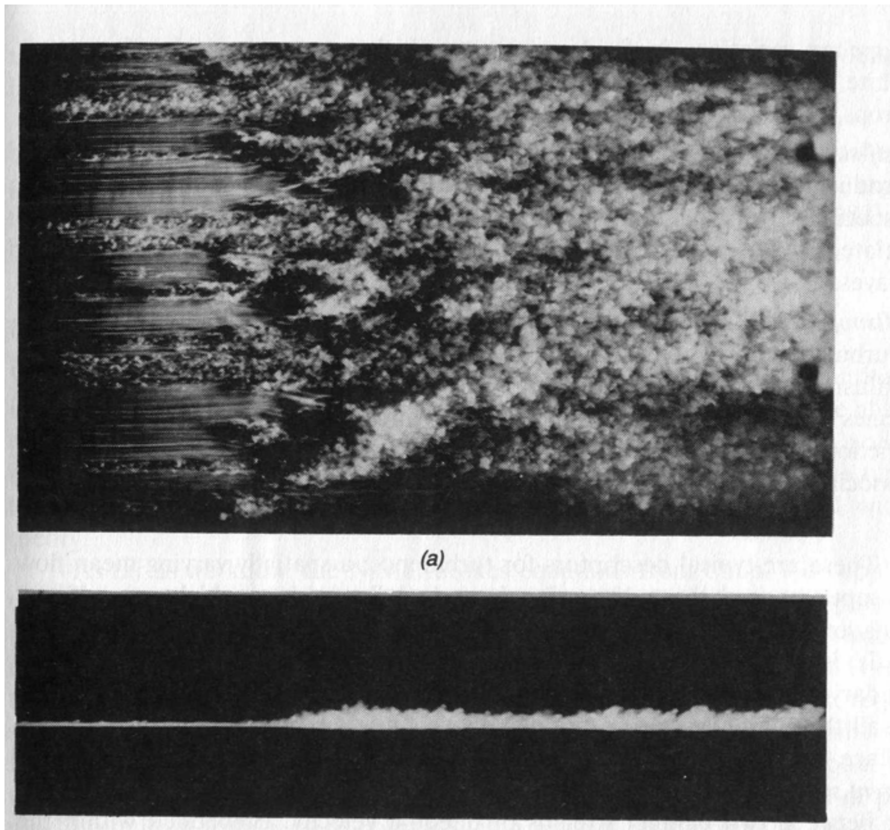
The Blasius solution and its derivatives are shown below.



Experimental laminar flat plate boundary layer data are plotted below:

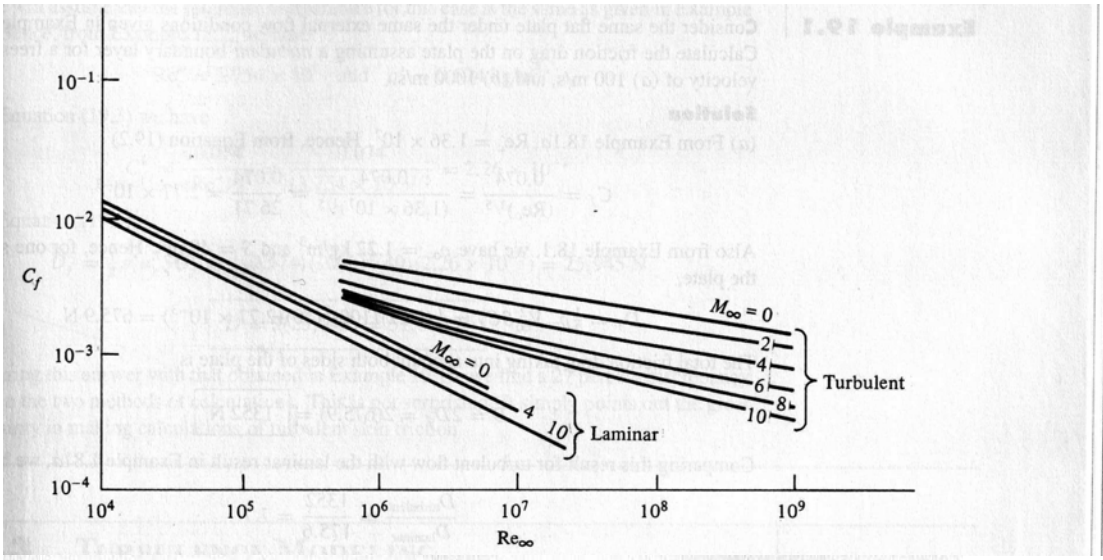


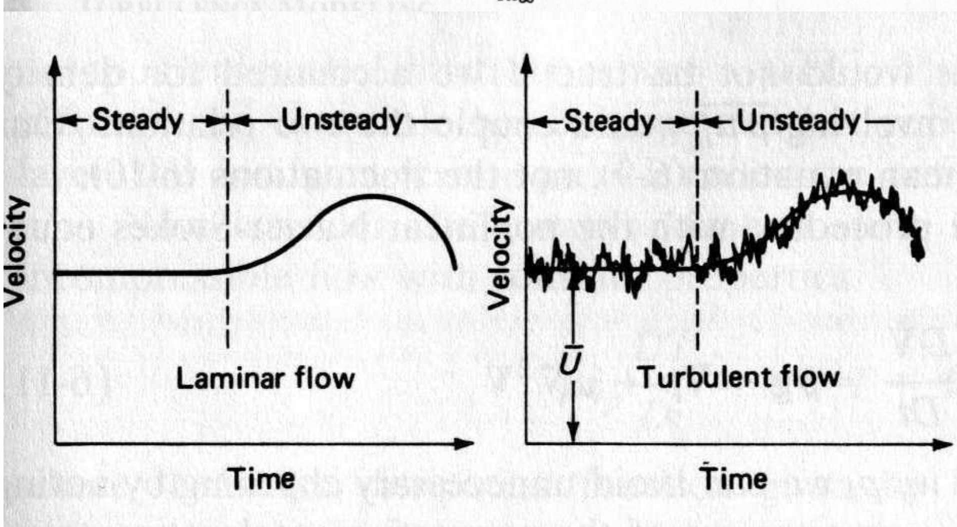
Turbulent Flat Plate Solution



Laminar flow is characterized by a smooth, layered appearance (laminate). Flows generally behave this way when the Reynolds number is low. As the Reynolds number increases the flow transitions from a smooth state to one with random perturbations about some mean. The figure above depicts experiments in which a flat plate boundary layer transitions from laminar to turbulent flow.

Characteristics of turbulent flow include fluctuations in pressure, temperature and velocity superimposed about a mean value. Note the thickening of the boundary layer in this case. These behaviors manifest themselves as a significant increase in skin friction as the flow transitions from laminar to turbulent and a reduced slope in skin friction decay with x, as illustrated in the figures shown next.





Some simple formulae for turbulent flat plate flows are

$$\delta \approx \frac{0.37x}{\mathrm{Re}_x^{1/5}} \tag{19.1}$$

$$C_f \approx \frac{0.074}{\mathrm{Re}_c^{1/5}} \tag{19.2}$$

$$C_f \approx 0.246\,\mathrm{Re}_\theta^{-0.268}\,10^{-0.678H} \tag{19.3}$$

$$C_f \approx 0.058\,\mathrm{Re}_\theta^{-0.268}\left(0.93 - 1.95\log_{10} H\right)^{1.705} \tag{19.4}$$

The first feature of Eq. (19.1) is the fact that turbulent boundary layers grow more quickly than laminar:

$$\delta \propto x^{4/5} \qquad \text{turbulent flows}$$

$$\delta \propto x^{1/2} \qquad \text{laminar flows}$$

Entire careers have been and are now devoted to the study of turbulent flows, as such, we can only touch briefly on some important first topics.

Turbulent Velocity Profiles

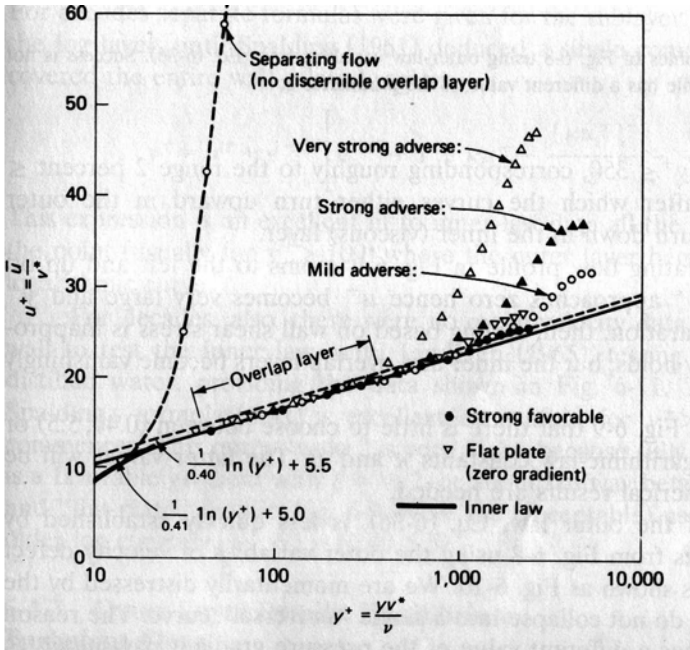
Dimensionless turbulent velocity profiles often correlate with the introduction of several new scaling parameters:

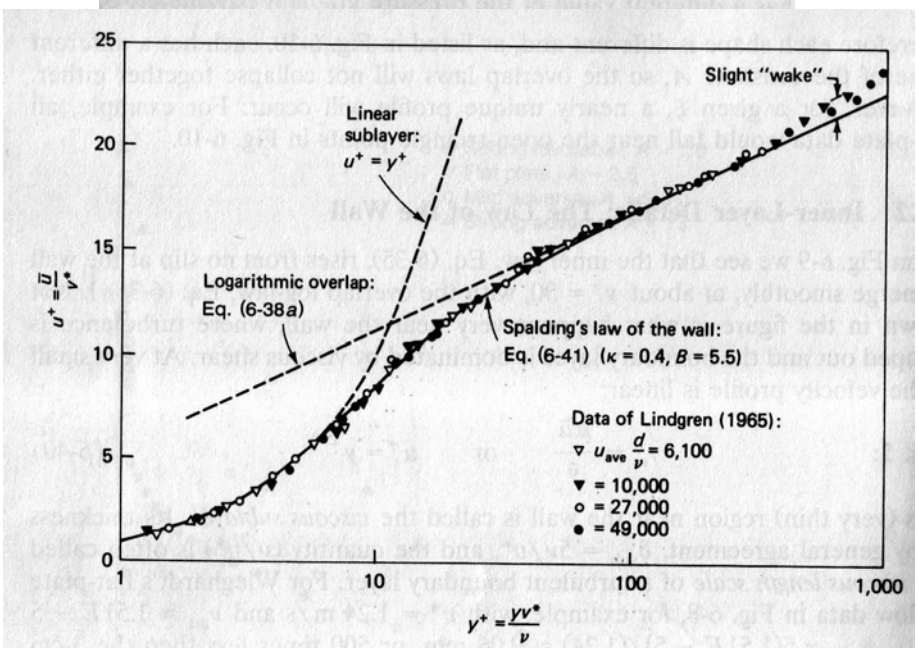
$$v^* = \sqrt{\frac{\tau_W}{\rho}} \qquad \text{wall friction velocity} \qquad (19.5)$$

$$u^+ = \frac{\overline{u}}{v^*} \qquad \text{turbulent inner-law velocity} \qquad (19.6)$$

$$y^+ = \frac{y v^*}{\upsilon} \qquad \text{turbulent inner-law wall distance} \qquad (19.7)$$

These variables are considered important because they scale the experimental data nicely.





<u>Linear Sublayer</u>

The linear or laminar sublayer is defined as $y^+ < 10$ we find

$$u^+ = y^+ \qquad (19.8)$$

Law of the Wall/Logorithmic Region

$$u^{+} = \frac{1}{\kappa}\ln y^{+} + B \tag{19.9}$$

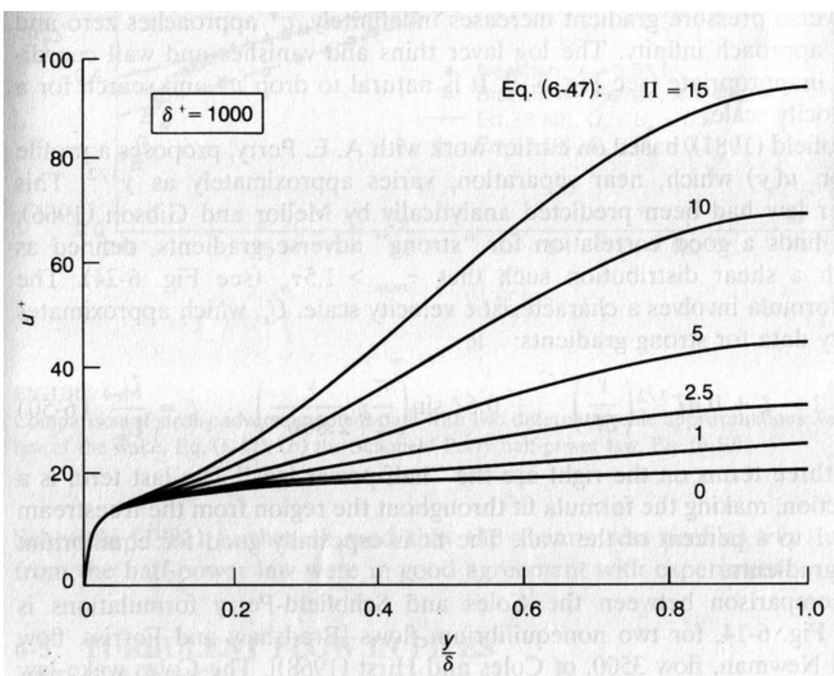
where $\kappa \approx 0.41$ and $B \approx 5.0$ are the generally accepted values of the constants.

Wake Region

The wake region becomes a much more difficult region to characterize as it is strongly dependent on external flow conditions like the pressure gradient. One modification of Eq. (19.9) that can be used in this region is

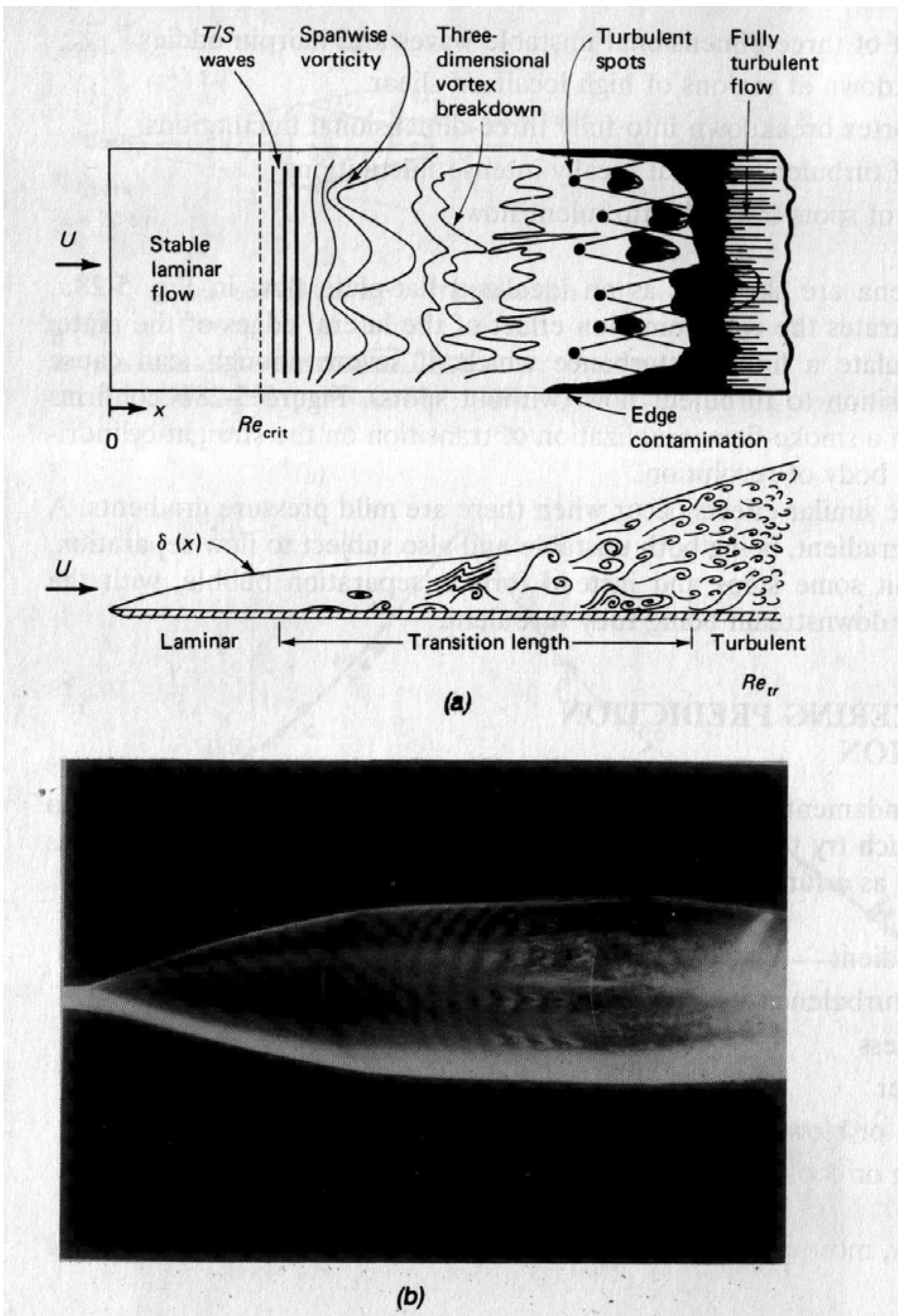
$$u^{+} = \frac{1}{\kappa}\ln y^{+} + B + \frac{2\Pi}{\kappa} f\left(\frac{y}{\delta}\right) \tag{19.10}$$

where Π is called the *Coles Wake Parameter*.

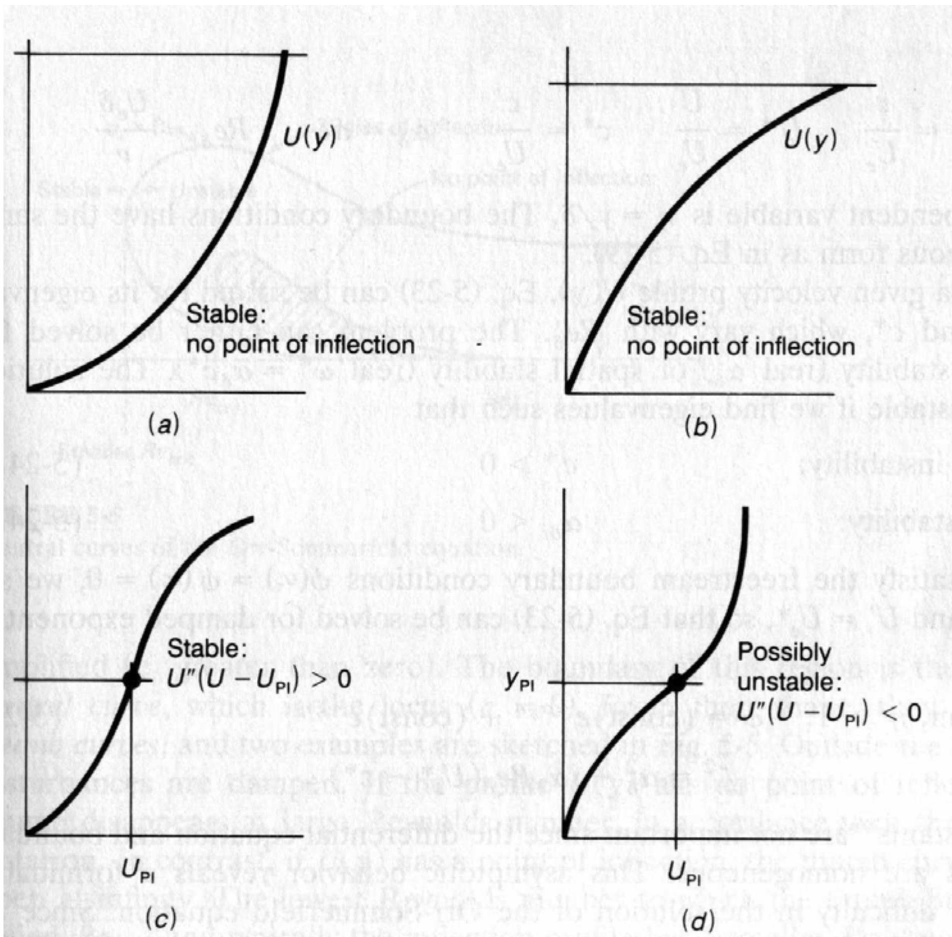


Transition to Turbulence

The transition from laminar to turbulent flow is often studied through the use of stability theory, i.e., the tendency of the flow to cause a disturbance to grow or decay. However, transition is much more involved as many events occur in the course of the transition process.



The stability of boundary layers is often tied to the velocity profiles. Inflection points in these profiles are often an indication of instability.



The N_{crit} variable used in XFOIL is a method in the class of e^N methods for predicting instability and the transition to turbulence. They are notoriously inaccurate and must be used with caution and more than a little physical insight. The XFOIL manual describes some potential choices for N_{crit} and a suggestion given for typical airfoil problems. However, be careful to check the definitions for everything used in XFOIL as they may be nonstandard!!!!

<u>Momentum Integral Relation</u>

A compelling set of methods for computing both laminar and turbulent boundary layers is found from the *Momentum Integral Relation*. This relation is found by starting with the boundary layer equations, multiplying continuity by $\left(u - U_e\right)$ and then subtracting the result from the momentum equation, this leaves:

$$-\frac{1}{\rho}\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial t}\left(u_e - u\right) + \frac{\partial}{\partial x}\left(u u_e - u^2\right) + \left(u_e - u\right)\frac{du_e}{dx} + \frac{\partial}{\partial y}\left(v u_e - v u\right) \tag{19.11}$$

Eq. (19.11) is then integrated from the wall to infinity.

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial t}\int_0^\infty \left(u_e - u\right)dy + \frac{\partial}{\partial x}\int_0^\infty \left(u u_e - u^2\right)dy + \frac{du_e}{dx}\int_0^\infty \left(u_e - u\right)dy - v_w u_e \tag{19.12}$$

This is called the *Karman Integral Relation*, which can be rewritten:

$$\frac{\tau_w}{\rho u_e^2} = \frac{C_f}{2} = \frac{1}{u_e^2}\frac{\partial}{\partial t}\left(u_e \delta^*\right) + \frac{d\theta}{dx} + \left(2\theta + \delta^*\right)\frac{1}{u_e}\frac{du_e}{dx} - \frac{v_w}{u_e}$$

(19.13)

for steady flow with an impermeable wall

$$\frac{C_f}{2} = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{u_e}\frac{du_e}{dx} \tag{19.14}$$

which states that given a δ^* and θ distribution, C_f can be determined. It is made particularly useful by finding a suitable velocity profile and/or correlation with experimental data.

Thwaites Method

Thwaites rewrote the momentum integral relation using a new parameter

$$\lambda = \frac{\theta^2 u_e'}{\upsilon} = \left(\frac{\theta}{\delta}\right)^2 \Lambda \tag{19.15}$$

The momentum integral relation when multiplied by $u_e\theta/\upsilon$ becomes:

$$\frac{\tau_w\theta}{\mu u_e} = \frac{u_e\theta}{\upsilon}\frac{d\theta}{dx} + (2 + H)\frac{\theta^2 u_e'}{\upsilon} \tag{19.16}$$

Thwaites idea was to group these terms and seek appropriate correlations with experimental data.

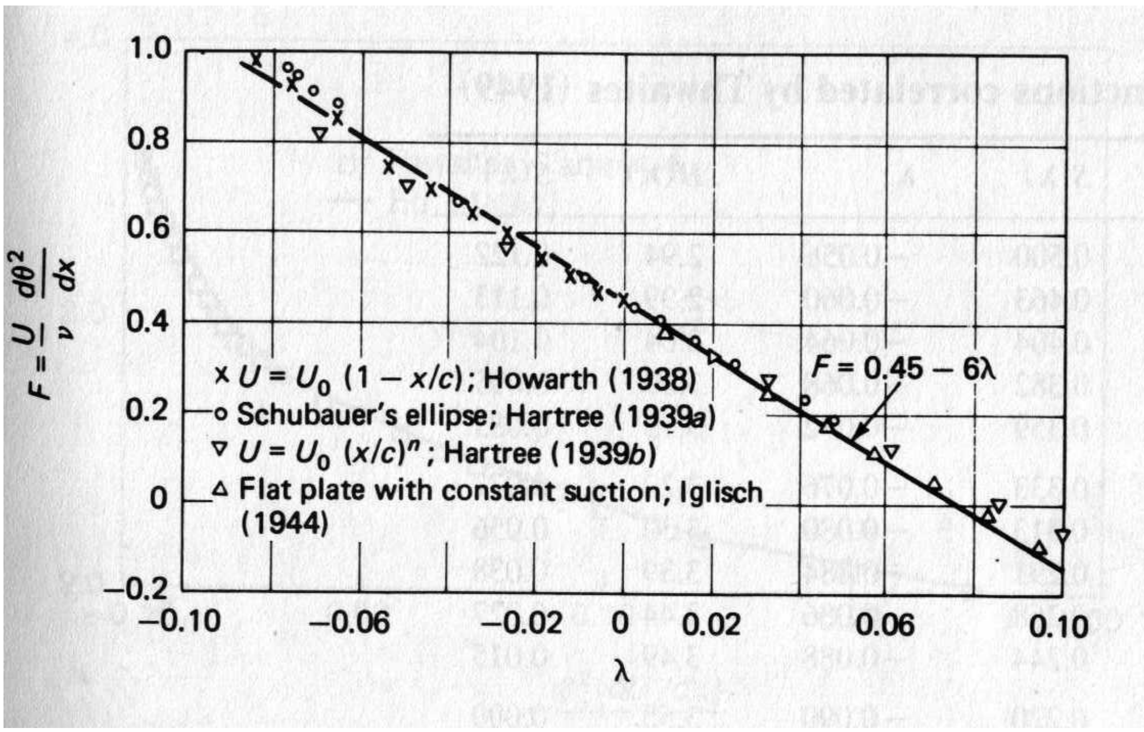
$$\frac{\tau_w\theta}{\mu u_e} \approx S(\lambda) \qquad \text{shear correlation}$$

$$H = \frac{\delta^*}{\theta} \approx H(\lambda) \qquad \text{shape-factor correlation}$$

Equation (19.16) then becomes

$$u_e \frac{d}{dx}\left(\frac{\lambda}{u_e'}\right) \approx 2[S(\lambda) - \lambda(2+H)] = F(\lambda) \qquad (19.17)$$

The data all fall along a single line which is astounding



Thwaites proposed

$$F(\lambda) \approx 0.45 - 6.0\lambda \qquad (19.18)$$

Given this form the momentum integral relation has a closed form solution:

$$\theta^2 \approx \frac{0.45\upsilon}{u_e^6}\int_0^x u_e^5 dx \qquad (19.19)$$

$$\tau_w = \frac{\mu u_e}{\theta} S(\lambda)$$
$$\delta^* = \theta H(\lambda) \tag{19.20}$$

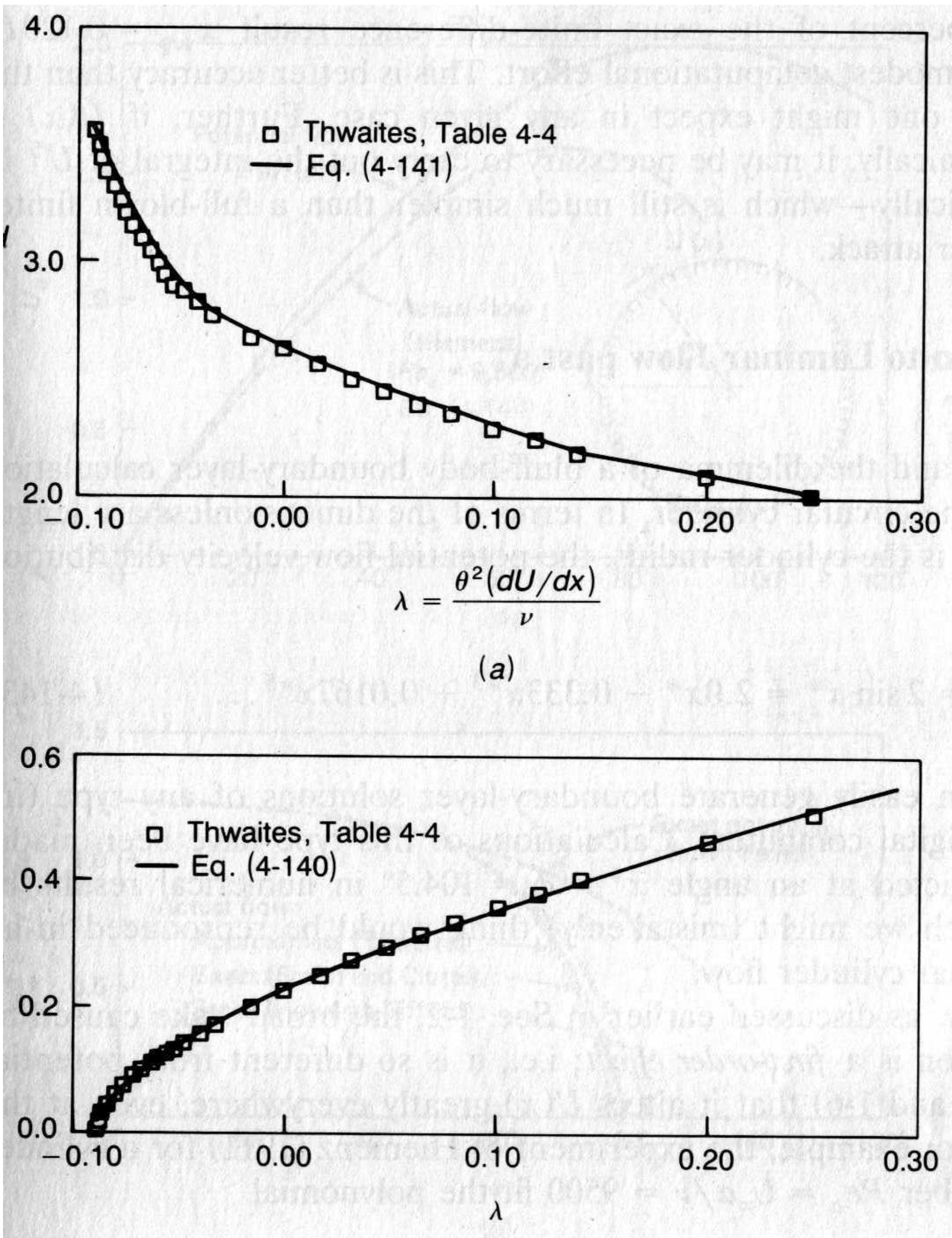
Thwaites further suggests

$$S(\lambda) \approx (\lambda + 0.09)^{0.62} \tag{19.21}$$

and

$$H(\lambda) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5 \tag{19.22}$$

where $z = (0.25 - \lambda)$.



Thwaites, Table 4-4
Eq. (4-141)
4.0
3.0
2.0
−0.10
0.00
0.10
0.20
0.30
$\lambda = \frac{\theta^2(dU/dx)}{\nu}$
(a)
Thwaites, Table 4-4
Eq. (4-140)
0.6
0.4
0.2
0.0
−0.10
0.00
0.10
0.20
0.30
λ

λ	$H(\lambda)$	$S(\lambda)$	λ	$H(\lambda)$	$S(\lambda)$
+ 0.25	2.00	0.500	−0.056	2.94	0.122
0.20	2.07	0.463	−0.060	2.99	0.113
0.14	2.18	0.404	−0.064	3.04	0.104
0.12	2.23	0.382	−0.068	3.09	0.095
0.10	2.28	0.359	−0.072	3.15	0.085
+0.080	2.34	0.333	−0.076	3.22	0.072
0.064	2.39	0.313	−0.080	3.30	0.056
0.048	2.44	0.291	−0.084	3.39	0.038
0.032	2.49	0.268	−0.086	3.44	0.027
0.016	2.55	0.244	−0.088	3.49	0.015
0.0	2.61	0.220	−0.090	3.55 (Separation)	0.000
−0.016	2.67	0.195			
−0.032	2.75	0.168			
−0.040	2.81	0.153			
−0.048	2.87	0.138			
−0.052	2.90	0.130			

Thwaites is applied once $u_e(x)$ is known. The solution process is:

1. Compute θ from Eq. (19.19)
2. Compute λ from Eq. (19.15)
3. Compute $S(\lambda)$ from Eq. (19.21)
4. Compute τ_w from Eq. (19.20a)
5. Compute $H(\lambda)$ from Eq. (19.22)
6. Compute δ^* from Eq. (19.20b)

Separation can be predicted by using $\tau_w = 0$ which implies

$$S(\lambda) = 0 \Rightarrow \lambda = -0.09$$