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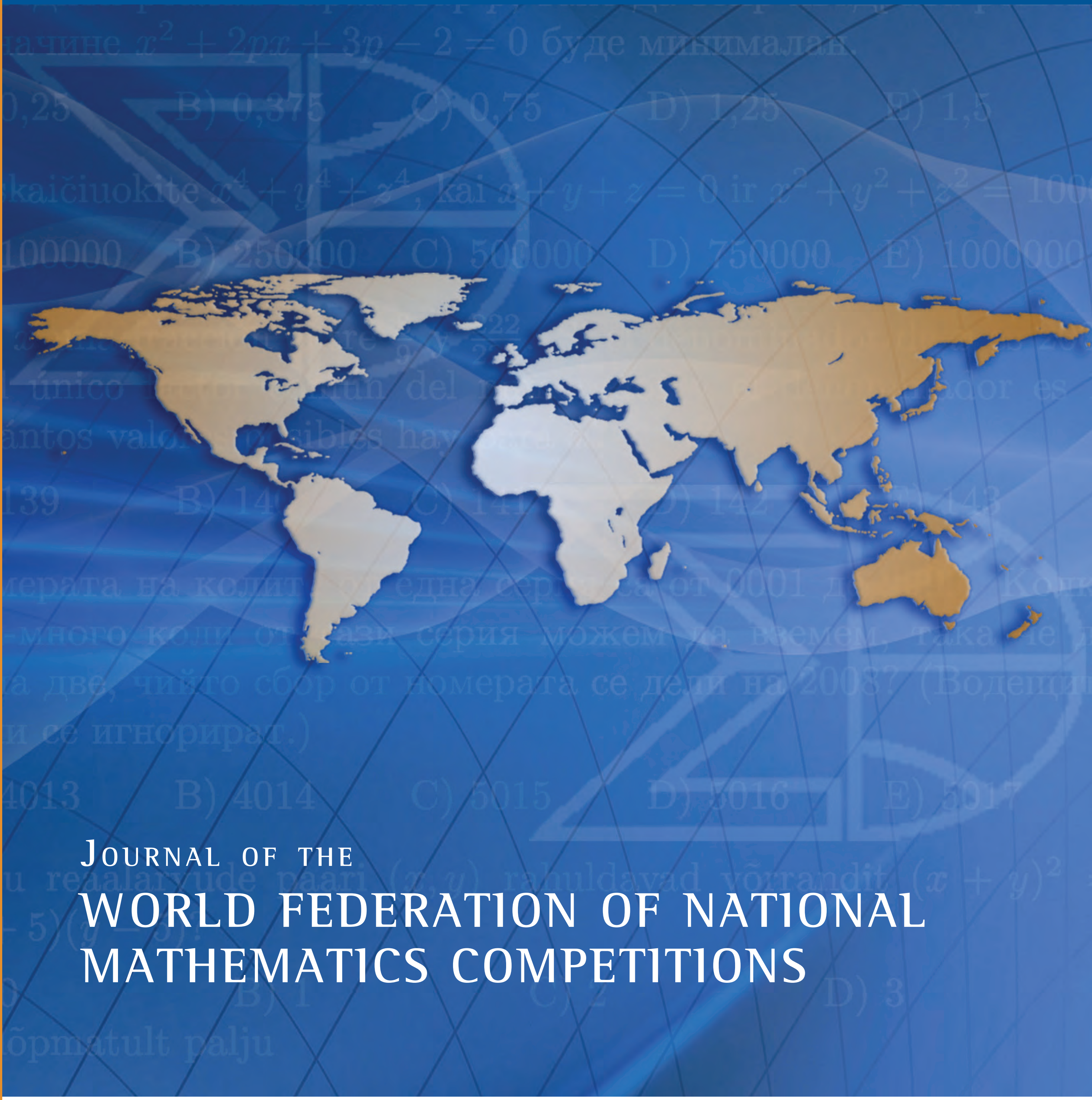
MATHEMATICS COMPETITIONS



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AUSTRALIAN MATHS TRUST

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World Federation of National Mathematics Competitions

Executive

Editorial Board

For WFNMC Standing Committees please refer to About WFNMC section of the WFNMC website http://www.wfnmc.org/.

From the President

Dear readers of *Mathematics Competitions* journal!

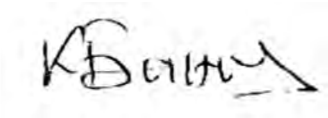
The global pandemic caused by the coronavirus disease has changed the life of the Planet and this has affected the mathematics education community. As you know, the most significant event, the 14th International Congress of Mathematics Education (ICME-14) that had expected to be held in 2020 was postponed by one year. This led to a change in the activities of the International Commission on Mathematical Instruction (ICMI). Traditionally the ICMI General Assembly takes place the day before the opening of ICME, which was not possible in 2020. Instead, the GA took place via virtual meeting in July 2020. The 2021-2024 ICMI Executive Committee was also elected online. Its members are: Frederick Leung (President), Jean-Luc Dorier (Secretary General), Merrilyn Goos (Vice President), Anjum Halai (Vice President), Marta Civil, Patricio Felmer, Mercy Kazima, Núria Planas, and Susanne Prediger. More information on each can be found here:
https://www.mathunion.org/icmi/organization/icmi-executive-committee/icmi-ec-2021-2024

The newly elected EC took office at the beginning of 2021. One of the decisions of the EC is to renew the communication with the ICMI Affiliate Organizations (one of which is WFNMC), by assigning a liaison person for each organization. The liaison person for WFNMC is Jean-Luc Dorier.

Let me go back to ICME-14. In November 2020 it was announced that ICME-14 will be held in a hybrid mode, i.e., it will be held simultaneously in face-to-face form (in Shanghai) as well as in an online form. The dates of the Congress are from July 11 till July 18, 2021. I hope that the scientific program of ICME-14 will remain the same. For now, Topic Study Group TSG-46 Mathematical Competitions and Other Challenging Activities at ICME-14 is designed to gather participants who are interested in the specific features, trends and needs of Competition Mathematics. This is the place where the WFNMC members may bring ideas for the discussions. I strongly advise you to regularly check the ICME-14 website (www.icme14.org) for updated information about the Congress.

The COVID-19 pandemic has also changed the life of the Federation. The WFNMC Executive decided to cancel the traditional one-day mini-conference of WFNMC that usually takes place the day before the opening of ICME. The reason is that it is not worth organizing the mini-conference in Shanghai, since it is not expected that many people will be traveling there. Making an online mini-conference does not seem sensible, because the idea for the mini-conferences is that traveling for ICME is a good reason for people to meet face-to-face for direct interaction. If we need an online conference, it can be organized independently from ICME. So, keep your bright ideas for presentations for the next meeting, the 9th WFNMC Conference that is planned to take place in July 2022 in Bulgaria.

My best regards,



Kiril Bankov
President of WFNMC
February 2021

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

It is both my pleasure and a challenge to take over the edition of this journal which deals with subjects so close to my endeavors of most of my life. I take the task with great help from my mother Mary Falk de Losada who has contributed greatly to the WFNMC even to the point of editing *Mathematics Competitions* for a year while an editor was found. Most of this issue was put together by her and I have mostly just concerned myself with the editing. I am very grateful for her great support!

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or Microsoft Word, accompanied by another copy in pdf. Texts in Word should mainly contain non-mathematical text and any images used should be sent separately. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is needed in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in BOLD and Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Olympiad problems of outstanding beauty (II)

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Krzysztof Ciesielski works at the Jagiellonian University in Kraków, from which he obtained his PhD. For nine years he was a vice-Head (responsible for teaching) of Mathematics Institute of this University. His mathematical specialty is topological dynamical systems. He works very actively in raising public awareness of mathematics. He is an author and co-author (mainly with Zdzis³aw Pogoda) of some books popularizing mathematics that were bestsellers in Poland and were awarded prestigious rewards, including the Steinhaus Prize and ”the Golden Rose Prize” for the best book popularizing science published in Poland. Since 1987 he has been a member of the Editorial Board (a Correspondent) of The Mathematical Intelligencer, in 1999 - 2012 he was an Associate Editor of The European Mathematical Society Newsletter, since 1996 he has been a member of the Editorial Committee of Polish monthly Delta (since 2003 being the vice-Chair of this Committee) and since 2007 he has been a member of the Editorial Board of the journal Wiadomœci Matematyczne of the Polish Mathematical Society, in 2018 he undertook a duty of being Editor-in-Chief of this journal. He is also involved in organization of several mathematical competitions, including Mathematical Kangaroo and Polish Mathematical Olympiad for secondary school students. Since 1980 he has been a member of the Kraków Committee of Polish Mathematical Olympiad and since 2008 he is the Chair of this Committee. He is also a member of the Editorial Board of the journal *Mathematics Competitions*.

Abstract
In the previous issue of *Mathematics Competitions* in the article on the session devoted to beautiful olympic problems (organized at the Congress of the Polish Mathematical Society) the problems given during this session were stated. In this article, the solutions of those problems are presented and their sources provided.

In September 2019 the Jubilee Congress of Polish Mathematicians for the 100th Anniversary of the Polish Mathematical Society took place in Kraków. During the Congress a special session *Olympiad problems of outstanding beauty* was organized. Several mathematicians presented there some problems (with solutions) they liked very much. More information of the session may be found in the first part of this article ([1]).

Here, the problems presented during the session are recalled and the ideas of the solutions are given[2]. The solutions were generally prepared by those who recounted the problems during the sessions. In the end, the information of the sources of the problems is provided.

[2]A modified version of this article was published in Polish language in the journal of the Polish Mathematical Society *Wiadomości Matematyczne*.

Problems and solutions

Problem 1 (Ryszard Rudnicki).

Assume that a polyhedron P is circumscribed about a sphere and it is possible to paint each face of P red or blue in such a way, that each two faces with a common edge are of different colors. Prove that the sum of the areas painted blue is equal to the sum of the areas painted red.

Take any face of P and draw segments joining the point of tangency between this face and the inscibed sphere with all vertices of this face. We have constructed triangles with a common vertex. Two triangles on different faces with a common base are painted in different colors. Consider such pair of triangles, say ABM and ABN, with the common base AB. Since the segments AM and AN are tangent to the sphere, they have the same lengths. Analogously, the lengths of BM and BN are the same. All three sides of the triangle ABM have the same lengths as the corresponding sides of the triangle ABN, so the triangles ABM and ABN have the same areas.

Therefore we expressed each face of P as the union of triangles, whereas those triangles may be paired up in such a way, that in each pair there is a blue triangle and a red triangle with the same areas. Now the equality of the sums of areas of suitable faces easily follows.

Problem 2 (Michał Wojciechowski).

Baron Münchhausen says that pines and birches grow in his magic forest and in the distance 1 kilometer from each pine there are precisely 10 birches. May baron Münchhausen say the truth?

Note that $65^2+0^2 = 65^2$, $39^2+52^2 = 65^2$ and $25^2+60^2 = 65^2$. Of course, $(\pm 65)^2+0^2 = 65^2$, $(\pm 39)^2+(\pm 52)^2 = 65^2$ and $(\pm 25)^2+(\pm 60)^2 = 65^2$ as well. From these equalities it follows that each circle of radius 65 centred in a point $(2k+1, 2n)$, where k and n are integers, contains precisely 10 points with integer coordinates. An analogous property holds for circles centred in points $(2n, 2k+1)$.

If we plant birches in points $(2k+1, 2n)$ and $(2n, 2k+1)$ for $k = 1, 2, \ldots, N$, $n = 0, 1, \ldots, N+1$, and pines in points $(2k, 2m)$, where $-4 \leq k \leq N+4$, $-4 \leq m \leq N+4$, then for N sufficiently large (after a suitable change of scale) we make a magic forest.

Problem 3 (Edward Tutaj).

A sequence $(x_n)_{n=1}^{\infty}$ is given by a recurrent relation

$$x_{n+3} = x_n + x_{n+1} \cdot x_{n+2}$$

with the initial conditions $x_1 = x_2 = x_3 = 1$. Prove that for each positive integer p some multiple of p is a term of (x_n).

Fix a $p \in \mathbb{N}$. We prove that there exists such $n \in \mathbb{N}$ that p is a divisor of x_n. For $p = 1$ the property is obvious.

Assume that $p > 1$. Let r_n be the reminder after division x_n by p, i.e. $r_n \equiv x_n \bmod p$ and $0 \leq r_n < p$. Thus the sequence $(r_n)_{n=1}^{+\infty}$ satisfies the formula $r_{n+3} = r_n + r_{n+1} \cdot r_{n+2}$ (here the addition is understood as the addition of reminders) and the condition $r_1 = r_2 = r_3 = 1$ (as $p > 1$). We show that there exists an $n \in \mathbb{N}$ with $r_n = 0$.

We have $r_n = r_{n+3} - r_{n+1} \cdot r_{n+2}$, so the sequence $(r_n)_{n=1}^{+\infty}$ may be extended from $\mathbb{N}$ to $\mathbb{Z}$ in a unique way. Moreover, from the Dirichlet pigeon principle it follows that the sequence $(r_n)_{-\infty}^{+\infty}$ is periodic. Indeed, if we know three consecutive terms r_s, r_{s+1}, r_{s+2} of this sequence, we can determine all the terms. However, the set of all threesomes (u_1, u_2, u_3) with $0 \le u_i < p$ is finite, and the set of all threesomes $T_s = (r_s, r_{s+1}, r_{s+2})$ is infinite, so we can find such $k, l \in \mathbb{N}$ that $T_k = T_l$. To finish the proof it is enough to note that $r_0 = 0$.

The property of sequences given by special recurrent relations was probably first noticed for the Fibonacci sequence.

Problem 4 (Barbara Roszkowska-Lech).

Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\dfrac{a^2 + b^2}{ab + 1}$ is the square of an integer.

Consider the equation $x^2 + y^2 = k(xy + 1)$ with variables x and y. If (x, y) satisfies this equation, then (y, x) satisfies it as well. Let x be the smallest integer such that there exists an integer y which satisfies the equation $x^2 + y^2 = k(xy + 1)$. Of course $x \le y$.

We can write this equation as a quadratic equation $t^2 - kxt + x^2 - k = 0$ with variable t. The number y is a solution of this equation, let z be the second solution. Consider two cases:

(i) $z > 0$. Since $yz = x^2 - k$ and $x \le y$, we have $xz \le yz = x^2 - k < x^2$, so $z < x$. That is a contradiction.

(ii) $z \le 0$. Then from Viète's formulas ($yz = x^2 - k$, $y + z = kx$) we get

$(y + 1)(z + 1) = x^2 - k + kx + 1 = x^2 + (x - 1)k + 1 \ge 1.$

If y and z are integers, $y > 0$ and $z \le 0$, then the inequality $(y + 1)(z + 1) \ge 1$ is fulfilled only for $z = 0$. Thus $x^2 - k = 0$, so k is the square of an integer. This finishes the proof.

Note that such integers a and b that $k = \frac{a^2+b^2}{ab+1}$ is integer, exist. It is enough to take $b = a^3$ for any integer a. Then $k = a^2$.

If a pair (a, b) with $a < b$ fulfills the conditions of the problem, then the pair $(kb - a, b)$ fulfills them as well. Applying symmetry, in this way we can construct an infinite sequence of pairs fulfilling these conditions.

Problem 5 (Jakub Węgrecki).

Each positive integer was painted in one of k colours. Show that there exist four pairwise distinct integers a, b, c, d painted in the same colour satisfying the conditions:

$$ad = bc, \quad \frac{c}{a} = 2019^n, \quad \frac{b}{a} = 2020^m$$

for some positive integers m, n.

We first recall a known property

Consider Cartesian coordinate system with any point with positive integer coordinates painted in one of k colours. Then there exists a rectangle with sides parallel to coordinate axis with all the vertices painted in the same colour. $(*)$

Now assume that each positive integer is painted in one of k colours. Consider Cartesian coordinate system where each point with positive integer coordinates (x, y) is painted in the same colour as $2019^x \cdot 2020^y$.

Applying $(*)$ we conclude that there exists a rectangle with all the vertices painted in the same colour. Denote its vertices by (x_1, y_1), (x_2, y_1), (x_1, y_2), (x_2, y_2). Then $a = 2019^{x_1} \cdot 2020^{y_1}$, $c = 2019^{x_2} \cdot 2020^{y_1}$, $b = 2019^{x_1} \cdot 2020^{y_2}$, $d = 2019^{x_2} \cdot 2020^{y_2}$ fulfill the conditions of the problem,

Problem 6 (Michał Krych).

Show that if integers a, b fulfill the equation $2a^2 + a = 3b^2 + b$, then $a - b$ and $2a + 2b + 1$ are the squares of integers.

We have $b^2 = 2a^2 + a - 2b^2 - b = (a - b)(2a + 2b + 1)$. If a prime number p divides $a - b$ and $2a+2b+1$, it divides also b^2, so it divides b and $\big(2a+2b+1-2(a-b)\big)-4b = 4b+1-4b = 1$, which is impossible. By the equality $b^2 = (a-b)(2a+2b+1)$ we conclude that each prime appears in the factorization of $a - b$ the same number of times as in the factorization of b^2, so it is raised to an even power. The same is applicable to the factorization of $2a + 2b + 1$. Have we finished the proof?

By all means, not! Right here the hardest part of the problem appears. We still need to prove that $a - b > 0$. To prove this we must use the assumption that a are b integers, as it is easy to find such real numbers a, b that $a < b$ and $2a^2 + a = 3b^2 + b$ (for example, take $b = 1$ and $a = -\frac{1}{4}(1 + \sqrt{33})$). This may be shown by the analysis of divisibility of numbers.

One may ask about the number of solutions of this equation in integers. There are infinitely many of them, which follows from Pell's equation.

Problem 7 (Michał Krych).

Assume that six points are given on edges of a tetrahedron $A_1A_2A_3A_4$, each one on a different edge. For each vertex of this tetrahedron we take a sphere containing this vertex and these three given points that are contained on the edges which have this vertex as an endpoint. Prove that those four spheres has a nonempty intersection.

This problem was given to the participants of the final of one of Polish Mathematical Olympiads. In the same year, in a semifinal the participants of the competition were asked to solve an analogous planar problem:

Let P, M, N be points on the sides AB, BC, CA, respectively, of the triangle ABC, different from the vertices of this triangle. Prove that the circumcircles of the triangles ANP, BPM and CMN have a common point.

This two-dimensional version is not difficult, but the analysis of many cases is necessary and this was the reason that many participants of the competition had serious problems in solving this problem.

The three-dimensional version may by easily reduced to the planar case. We may assume that the intersection of each two spheres is a circle (to omit some singular cases). Denote by B_{ik} the given point contained on the edge A_iA_k ($i, k \in \{1, 2, 3, 4\}$, $i < k$) and by S_i the sphere containing A_i and the points B_{ij} or B_{ji}, $j \in \{1,2,3,4\}$, $i \neq j$. Let C_i be the point of intersection of the plane $A_jA_kA_\ell$ ($\{j,k,\ell\} = \{1,2,3,4\} \setminus \{i\}$) and spheres S_j, S_k, S_ℓ (the existence of such point C_i follows from the planar case). Now transform the sphere S_1 by stereographic projection from A_1 to the plane not containing A_1. The circles that are the intersections of the sphere with the planes $A_1A_2A_3$, $A_1A_3A_4$ and $A_1A_4A_2$ are transformed onto lines and circles $S_1 \cap S_i$ are transformed onto circles, which reduces the problem to the planar case.

Problem 8 (Bartłomiej Bzdęga).

We are given a convex pentagon $ABCDE$ with

$$AB = BC = CD, \qquad AE = EB = BD, \qquad AC = CE = ED.$$

Determine the measurements of its angles.

The equalities given in the assumptions hold if A, B, C, D are consecutive vertices of a regular dodecagon with the centre E.

Set $x = AB$, $y = AE$, $z = AC$ and $\varphi = |\sphericalangle BAC|$, $\psi = |\sphericalangle CAE|$. From the formula $\cos(\varphi + \psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi$ it follows that $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = 5$, analogously we get $\frac{x^2}{z^2} + \frac{z^2}{y^2} + \frac{y^2}{x^2} = 5$. From these equalities we conclude that two of three numbers x, y, z are equal. The pentagon is convex, so $y = z$. A simple calculation on the measurement of angles finishes the solution.

Another solution of this problem was recently presented to Bartłomiej Bzdęga by Tomasz Cieśla. Outside the pentagon $ABCDE$ we construct the triangles ABF and CDG which are congruent to the triangle BCE. Then the isosceles triangles ABC and EAF are similar, so the triangles BCD and EDG are similar as well. By suitable proportions we obtain the equality $|EF| = |EG|$, hence the triangles EBF and GCE are congruent. Again, to finish the solution it is enough to make a simple calculation on the measurement of angles.

Problem 9 (Dominik Burek).

The integers $a_1, a_2, \ldots, a_n$ satisfy the inequalities

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

Show that if m is the number of different prime divisors of the product $a_1a_2\ldots a_n$, then

$$(a_1a_2\ldots a_n)^{m-1} \geq (n!)^m.$$

Express numbers a_i as $a_i = p^{k_i} \cdot b_i$, where p does not divide b_i for any prime factor p of a number $a_1a_2\ldots a_n$. From the inequalities $a_1 < a_2 < \ldots < a_n < 2a_1$ it follows that b_i are pairwise different. Indeed, if $b_i = b_j$ for some $i < j$, then

$$\frac{a_j}{a_i} = \frac{p^{k_j} \cdot b_i}{p^{k_i} \cdot b_i} = p^{k_j - k_i} \geq 2,$$

which is impossible. Thus

$$b_1b_2\ldots b_n \geq n!.$$

If we multiply above inequalities for each prime p which divides the product $a_1 a_2 \dots a_n$ we get the desired conclusion.

Problem 10 (Andrzej Grzesik).

Determine the maximal number of lines in the three-dimensional space such that all of them have one point in common and the angles between each two of them are the same.

Denote the intersection point by P and assume that one of the lines is vertical. Consider a sphere centred in P. Each non-vertical line and the vertical line form congruent angles, so the points of intersection of all these lines and the sphere are contained in two antipodal circles parallel to the horizontal plane. We may repeat this reasoning for another line and conclude that the intersection points of other lines and the sphere also are contained in two antipodal circles. Two pairs of antipodal circles may intersect only in four pairs of antipodal points, which means that besides these two lines there are at most four other lines satisfying the conditions of the problem. We have shown that the upper bound for the number of lines is 6. To show that this upper bound is obtained it is enough to join antipodal vertices of regular icosahedron.

Problem 11 (Grzegorz Świątek).

A light ray moves in the region $U := \{(x, y) : x > 0, 0 < y < x^2\}$ *reflecting from the boundary components according to the rule of equal angles of incidence and reflection. Prove that its trajectory will reflect only a finite number of times.*

A full trajectory consists of the forward and backward part; however it suffices to consider the forward part, since the remainder can be treated by reversing the direction of the original motion.

Since the region above the parabola is convex, the reflections will alternate between the boundary components. Let (x_n) denote the x-coordinates of the points of reflection in the parabola numbered in the order they occur, and likewise $(\hat{x}_n)$ be the points of reflection on the x-axis and without loss of generality the motion is $\hat{x}_1 \to x_1 \to \hat{x}_2 \to \cdots$. Let α_n denote the measure of the angle $x_n \hat{x}_n 0$. For example, $x_n \geq \hat{x_n}$ if and only if $\alpha_n \geq \frac{\pi}{2}$.

From the inequality

$$\tan \frac{\alpha_{n+1}}{2} - \tan \frac{\alpha_n}{2} > 2x_n. \tag{1}$$

it follows that α_n form a strictly increasing sequence. One observes that once for some n we get $\alpha_n \geq \frac{\pi}{2}$, the sequence (x_n) increases from that point on, (α_n) tends to π which means that after finitely many steps it stops hitting the parabola. Thus, for the trajectory to reflect infinitely many times, $\alpha_n < \frac{\pi}{2}$ for all n, while sequences (x_n) and $(\hat{x}_n)$ are strictly decreasing. Then another simple geometric estimate is:

$$\text{If } x_n > x_{n+1}, \text{ then } x_n - x_{n+1} < \frac{2x_n^2}{\tan \alpha_{n+1}} \leq C x_n^2, \text{ where } C = \frac{2}{\tan \alpha_1} > 0. \tag{2}$$

If we now suppose that the claim is false, we get a sequence (x_n) which is strictly decreasing and must tend to 0, or (α_n) would tend to $+\infty$ by (1).

The key estimate is that

$$\sum_{n=1}^{+\infty}\left(1-\frac{x_{n+1}}{x_n}\right)=+\infty. \tag{3}$$

Otherwise for N sufficiently large we get

$$1>\sum_{n=N}^{+\infty}\left(1-\frac{x_n}{x_{n+1}}\right)>\sum_{n=N}^{+\infty}\frac{x_n}{x_N}\left(1-\frac{x_n}{x_{n+1}}\right)=x_N\sum_{n=N}^{+\infty}(x_n-x_{n+1})=\frac{x_N}{x_N}=1$$

which is a contradiction.

From (2) and (3) we conclude that $\sum_{n=1}^{+\infty}x_n=+\infty$, by (1) the sequence $\tan\frac{\alpha_n}{2}$ tends to $+\infty$ and α_n tends to π which implies the claim as already observed.

The problem of whether this result remains true when instead of x^2 we consider the graph of any smooth convex function, positive for $x>0$ and 0 at 0, as the upper boundary of U, seems to be still unsolved.

Problem 12 (Krzysztof Oleszkiewicz).

Assume that r is a positive integer. Show that for any real numbers $a_1,\ a_2,\dots,a_r$ the inequality

$$\sum_{m=1}^{r}\left(\sum_{n=1}^{r}\frac{a_ma_n}{m+n}\right)\geq 0.$$

holds. Determine for which numbers $a_1,\ a_2,\dots,a_r$ the equality occurs.

Consider polynomial functions defined by $P(x)=a_1+\sum_{m=2}^{r}a_mx^{m-1}$ and $Q(x)=\sum_{m=1}^{r}\left(\sum_{n=1}^{r}\frac{a_ma_n}{m+n}x^{m+n}\right)$. As $Q'(x)=xP(x)^2$, we have $Q'(x)\geq 0$ for $x\in[0,1]$. Therefore the function Q is increasing on the interval $[0,1]$ and

$$0=Q(0)\leq Q(1)=\sum_{m=1}^{r}\left(\sum_{n=1}^{r}\frac{a_ma_n}{m+n}\right),$$

which completes the proof of the inequality.

If the equality holds, then for $x\in[0,1]$ we have $0=Q(0)\leq Q(x)\leq Q(1)=0$, so $Q\equiv 0$ on $[0,1]$. Thus $Q'\equiv 0$ on $[0,1]$, and consequently $P\equiv 0$ on $(0,1]$. If P were a polynomial of positive degree, it would have a finite number of real roots. Therefore P must be the zero polynomial and $a_1=a_2=\ldots=a_r=0$. Of course, the equality holds for the zero polynomial.

This inequality is an elementary case of the known result from functional and harmonic analysis: the Hilbert transform is a positive semidefinite operator on the space of all sequences (x_n) such that

$$\sum_{n=1}^{+\infty}|x_n|^2<+\infty.$$

Problem 13 (Krzysztof Ciesielski).

Assume that each three out of six points in the plane are vertices of a scalene triangle. Prove that the shortest side of one of the triangles is at the same time the longest side of another.

Let us start from a well known problem,

Six points in the plane are given. Any three of them are vertices of a triangle. The segments joining these points are coloured, either green or blue. Prove that three of the given points are vertices of a triangle with sides painted with the same color

To prove this, select one point and consider all segments having this point as an endpoint. By the pigeonhole principle, at least three of them are the same colour, say green. Now consider the endpoints of these three green segments. If whichever pair is connected with a green segment, then we have a green triangle; if it isn't, then all of them are connected with blue colour and hence, the blue triangle exists.

Now come back to the original problem.

Let us colour the segments with vertices among the six given points either green or blue according to the following rule. Colour blue each segment for which there exists such a triangle that this segment is the shortest side of this triangle, and colour green the remaining segments. We know that a one-color triangle exists. Of course, any triangle cannot have all green sides, as the shortest side in this triangle is blue. So it is a blue-color triangle, and its longest side satisfies the conditions of the problem.

Sources of the problems.

Problem 1 is taken from the semifinal of 26th Polish Mathematical Olympiad in 1974/1975.

The author of Problem 2 is Fedor Nazarov. The problem is taken from the final of 54 Leningrad Mathematical Olympiad in 1988.

Problem 3 was long time ago presented to Edward Tutaj (as an especially attractive olympic problem) by Andrzej Mąkowski. Andrzej Mąkowski (1937–2007) was a legendary person in the history of the Polish Mathematical Olympiad and in the society of Polish mathematicians. He was one of the winners of 4th Polish Mathematical Olympiad (in 1953/54). Shortly after graduating mathematics at the University of Warsaw he joined to the Main Committee of the Polish Mathematical Olympiad, being its member since 11th Olympiad (1959/60) up to his sudden death in 2007. Since 2007 there is a prize, named by him, for the mostly elegant written solution during the final of each Polish Mathematical Olympiad. Mąkowski was a member of the Editorial Board of the Polish popular monthly *Delta* (the journal presents mathematics, physics, informatics and astronomy in a popular way) since the beginning of the monthly in 1974 up to his death, and since 1962 he was a member of Editorial Board of the journal of the Polish Mathematical Society *Wiadomości Matematyczne* (in 1965–1974 he was the secretary of the Board, in 1975–1979 and since 1995 up to his death he was Associate Editor). He was an author of several important research papers (mainly concerning number theory). He was known as a person with incredible good memory. Very frequently, meeting another mathematician, Mąkowski was saying this person about some new mathematical interesting news; Mąkowski knew pretty well what may interest a particular person. Mąkowski was several times the Team Leader of Polish team for the International Mathematical Olympiad.

Problem 4 is taken from 29 International Mathematical Olympiad in 1988.

Problem 5 was presented to Jakub Węgrecki by Marcin Radwański (a school teacher in Tarnów who prepares many secondary school students to Mathematical Olympiad) a few years ago, when Węgrecki was a school student. According to an olympiad tradition, the numbers in the problem were updated to the present years.

Problems 6 and 7 are from the final of 16th Polish Mathematical Olympiad in 1964/1965. Problem 7 was then solved by only one participant.

The author of Problem 8 is Bartłomiej Bzdęga, who presented it during the session. The problem was given to the participants of 7th Greater Poland Mathematical League (which is a competition for secondary school students in Poznań County area).

The author of Problem 9 is Dominik Burek, who presented it during the session. The problem was given to the participants of the final of 68th Polish Mathematical Olympiad in 2016/2017.

Problem 10 was given to the participants of the final of 55th Polish Mathematical Olympiad in 2003/2004.

The origin of Problem 11 is unknown, probably it is taken from an international olympiad from the 1970's.

Problem 12 was given to the participants of the final of 43rd Polish Mathematical Olympiad in 1991/1992 and Krzysztof Oleszkiewicz was its author.

Problem 13 was given to the participants of semifinal of 27th Polish Mathematical Olympiad in 1975/1976. The lemma on a monochromatic triangle which was used in the solution was given in the final of 17th Polish Mathematical Olympiad in 1965/1966, but this problem was known earlier. According to Soifer ([2]), in 1953 Frank Harary suggested it as a problem 18 for the W. L. Putnam Mathematical Competition, however Harary was not its author; he claimed that the problem has been around for a long time in mathematical folklore.

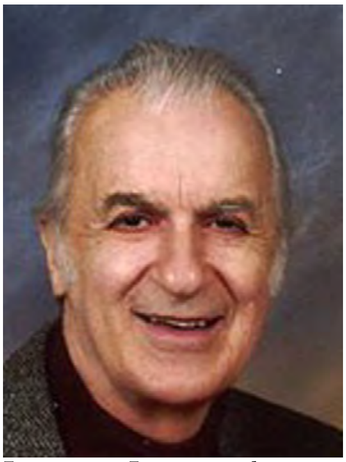
References

[1] K.Ciesielski, *Olympiad problems of outstanding beauty (I)*, Mathematics Competitions, 33(2020), No.2, 38-47.

[2] A.Soifer, *The Mathematical Coloring Book. Mathematics of Coloring and the Coloring Life of Its Creators*, Springer, New York 2009.

In memory of Professor Emerita Nura Turner (1905-1994)

George Berzsenyi



Dr. George Berzsenyi is Professor Emeritus of Rose-Hulman Institute of Technology in Terre Haute, IN, USA. He served on the USA Mathematical Olympiad Committee for 12 years, was the first chairman of the committee in charge of the American Invitational Mathematics Examination and served on the Australian Mathematics Competitions Problems Committee 4 times. He is the recipient of the WFNMC's Erdös Award (1996) and the MAA's Gung & Hu Award for Distinguished Service (2016).

Last year I gave a photograph to Joe Palca, who showcased my mentoring of students in a National Public Radio interview[3]. The picture showed Jimmy (now James) Wilson, one of my protegées, receiving an award from Nura Turner. Seeing the photo, Rachel Levy[4] of the Mathematical Association of America (MAA) contacted me, inquiring about Nura, since she was about to prepare an article about her. The purpose of Rachel's article[5], written jointly with Susan Kennedy[6] was to announce the fact that following Susan's research and convincing arguments, the Board of Directors of the MAA decided to rename the Greitzer - Klamkin Award given each year to the winner of the USA Mathematical Olympiad (USAMO), the Greitzer - Klamkin - Turner Award in recognition of Nura's pioneering work in the Olympiad program.

Unfortunately, Nura was no longer around to enjoy the recognition. However, she was present in 1978, when the MAA gave her, along with Sam Greitzer and Murray Klamkin the Certificate of Merit, a special award given at irregular intervals for some special work or service associated with mathematics or the wider mathematical community.

In response to Rachael's inquiry, I reminisced a bit about Nura and managed to correct the mistaken caption that went with her picture. She incorporated them into their article. Knowing Nura not only as a mathematician, but also as a family friend, I could not help reminiscing a bit more. Hence the present article, in which I will show a few pictures of Nura and say a few more words about her contributions. Moreover, noticing that at the time of her death nobody in the mathematical community said farewell to her, I also consider this writing as an epitaph to her memory.

[3] https://www.npr.org/2019/04/07/707326070/a-math-teachers-life-summed-up-by-the-\gifted-students-he-mentored?fbclid=IwAR3vPaVnk40jrt6zw8tjH1YTBLFQBRfk93BD38H6-\EPPS-hgFOEARqW7zRw

[4] Deputy Executive Director of the MAA; former professor and associate dean at Harvey Mudd College

[5] https://www.mathvalues.org/masterblog/maa-surfaces-an-amc-champion

[6] Executive Assistant, MAA

Nura Dorthea Rains Turner was born in McGovern, Iowa on March 5, 1905 and was named after her grandmother Nura Dorthea Rains, originally from the island of Malta. She received her degree in mathematics in 1928 from the University of Iowa. Later she also earned a master's degree and had additional graduate training at Brown University, Ohio State University, Virginia Polytechnic Institute and State University, the University of Florida and North Carolina State University. She worked as a statistician in industry until 1945 but she had started her career as a high school teacher.

In fact, her first teaching job was in Port Arthur, a neighboring city to Beaumont, Texas, where I lived. Therefore, when I invited her in 1978 to the Lamar Mathematics Day program that I initiated a year earlier, she happily accepted my invitation, along with an invitation by her former students to their 50th class reunion. That was the first time she visited us.

She stayed at our house for several days, enjoyed her students' 50th reunion and was well-received both by the students and the faculty at our Mathematics Day program, during which she talked about the USAMO and gave out some of the prizes. The picture above shows her during her lecture, with the heads of the computer science and the mathematics departments flanking her, while the ones below show Nura sitting with one of our visiting teachers and standing between my wife, Kay and myself.

Returning to the story of Nura, in 1946 she became an assistant professor of mathematics at the State University of New York in Albany. She taught there for the rest of her career. She was probably a strict but excellent teacher; I have no information about that. But I know that she was energetic, driven and most successful as a planner and organizer in the area of competitions for talented high school students. Moreover, she was a true visionary in recognizing and exploring the possibilities and was well-prepared to face and to prevail against the obstacles she encountered.

Thus, by the time she wrote her oft-quoted article asking, "Why can't we have a USA Mathematical Olympiad?"[7], she surveyed the possibilities and recognized that the time was ripe and the students were ready for such a competition. She also knew via her contacts that the German Democratic Republic (i.e., East Germany) was willing to invite the United States to the International Mathematical Olympiad (IMO) in 1974. Moreover, she knew from experience that our students were capable enough to hold their own against the competitors of other countries.

Nura's involvement with competitions dates back to the earliest "Mathematics Contest", which was first held in the Metropolitan New York area in 1950, and soon spread to the rest of the state. By 1958 it became national, and later became known as the American High School Mathematics Examination [8](AHSME). It spread beyond the borders of the US too. In England, where it was also administered, it was used to identify the top 60-70 students for the British Mathematical Olympiad (BMO). Nura followed the British example with interest and as the first contest chairperson for the Upstate New York Section of the MAA, she made arrangements to take 6 students from the Upstate New York area to London to compete against a similar group of students from there in the 4th BMO in 1968. The New Yorkers held their own, which convinced Nura that if there was a USA Mathematical Olympiad and if its winners competed against students from other countries in the International Mathematics Olympiad (IMO), the American team would hold its own too. Thus, in her famous article Nura advocated not only for the creation of a national Olympiad program in the USA, but also in favor of our participation in the IMO.

Few people remember that Nura's campaign for the USAMO started years before her article appeared. While she found some supporters, many believed that the US would

[7]The American Mathematical Monthly, 1971, pp.192-1955

[8]More recently, it split into the AMC-10 and AMC-12, which were joined by the AMC-8 for younger students.

embarrass itself in such a competition, that our students could not be prepared adequately and that the organization and grading of an Olympiad would be too much work. In her article she addressed the various objections based on facts rather than false beliefs, and hence it was decided to launch the USAMO in 1972. And indeed, the preparation, the administration, and the evaluation of the USAMO turned out to be relatively easy.

On the other hand, Nura gave herself a harder task, the organization of the ceremonies for the winners of the USAMO. She saw to it that the winners and their parents were welcomed at the National Academy of Sciences with proper formalities observed. There was a mathematical lecture to be delivered by a well-known mathematician for them as the audience, a visit to the National Science Foundation peppered with yet more talks, and visits either to the Capitol, the White House or the Library of Congress, to be followed by a formal Reception and Dinner at the Department of State's Diplomatic Reception Room. Nura Turner made sure that the winners of the USAMO were given a proper welcome by the governmental and the scientific community. As a member of the MAA committee in charge of the USAMO from 1977 to 1989, I was one of the hosts whenever I was able to attend the ceremonies. My wife accompanied me on three of those occasions too. Since Nura was of my mother's generation, I was one of the young ones at Nura's beck and call. She was a perfectionist with respect to every aspect of the program, as well as concerning the prizes given to the students. In particular, she managed to secure the support of IBM to cover the costs of the lavish celebrations and of the management of Hewlett-Packard, which gave some of the first programmable calculators as prizes to the winners year after year.

At this point I might mention that Nura shared with me a letter written to her by Murray Klamkin on March 13, 1972 in which Murray starts off saying: "Apparently Greitzer has been procrastinating on the problem of adequate awards for the winners of the Olympiad. And from his last letter, he seems to have a negative attitude about it." Murray goes on to say, "If in addition to my letter to him, you can 'light the fire under him', it would be greatly appreciated. One trouble with him is that he 'thinks too small'. Consequently, anything you can do to obtain suitable awards should be highly appreciated by our full committee."

In another letter to Nura, written on November 16, 1983 Murray wrote *I seem to recall that my presence as a new member of the committee finally tipped the balance towards getting the Olympiad (for) which you have worked so hard for a long time.*" He went on to say *Sam and I were on the Board of Governors when the USA participation in the IMO was discussed and again it was a close vote for participation. It was my feeling that the people against it were afraid of a poor USA showing and that it would reflect badly on mathematics teaching in the country.*"

Murray wondered if Nura had a copy of the memo in which Sam expressed his doubts about the Olympiad, and Nura must have complied by producing a letter written by Sam to Henry Alder, the national secretary of the MAA on April 28, 1972, in which Sam admitted that "*First, I myself had grave doubts about the advisability and feasibility of such an Olympiad." In the same letter, which was also shared with me by Nura, Sam also argued against awards for the winners by claiming that "These students take the Olympiad because they like mathematics.*"

Later in the letter[9], Sam also claimed that his attitude was shared by Julius Hlavaty[10] and Murray Klamkin. But he contradicts himself and says at the end of the letter that General Electric should be approached for substantial cash awards, say, \$20,000, which didn't happen. That turn-about was characteristic of Sam, who later became the strongest supporter of the USAMO. He became the chairman in charge of the program and the Leader of our team to the IMO with Murray serving as the Deputy Leader for more than a decade. I was on good terms with both of them, having had Murray as an invited speaker and a house-guest when I was teaching at Rose-Hulman, while Sam and I shared a room at the last Joint Meeting of the MAA and AMS (American Mathematical Society) he attended in January 1988. He died a month later and it was I who wrote (jointly with the late Walter Mientka) his obituary in the present publication[11].

Nura's continuing interest in the winners of the USAMO also lead to some articles she wrote, in which she reported on the whereabouts of the winners, the universities and the fields of studies they chose, their accomplishments on the Putnam Examination (a demanding mathematics competition at the university level), their progress towards higher degrees, choices of employment, as well as other events in their lives. She kept track of well over 100 of them in a USAMO Newsletter, which was co-edited with her by one of the former winners.

I still have a copy of its penultimate 1989 issue[12] that was co-edited by 1982 winner Tsz-Mei Ko, in which she reported on the whereabouts of the 108 former winners of the USAMO and summarized their accomplishments at the IMO as well. In a 'Commentary' on page 17, she was also making very strong arguments against present practices as of 1989 in the selection of the team members to the IMO. Since those practices did not change and since I totally agree with Nura's attitude, I quote below the first and the last paragraph of Nura's Commentary. I hereby call them to the attention of all readers, but especially to the Leaders and Deputy Leaders of teams to the IMOs:

"*Might consideration be given to selecting the members of our team only from the 8 USAMO winners that for reference I'll call Group 1, rather than from the membership of the short 3 or 4 week training program (MOP) that precedes our sending a team to the IMO that I will call Group 2? Supporters of Group 2 base their attitude on 'competing-to-beat', that is, competing with the purpose of achieving a total of the scores of members that would be greater than such totals of other teams. But that was not the aim of the "father" of the IMO, Tiberiu Roman, who as the one responsible for the IMO, organized the first competition held in Romania in 1959 at which only Eastern European countries were invited and 7 attended and organized it as a competition of individuals - not as one of teams.*"

At this point I might mention that I also met Tiberiu Roman in Bucharest in 1978, and he assured me of the same intentions as voiced by Nura, who went on in the next two

[9] which I will deposit - along with the other two - in the MAA's Archives in Austin, TX.
[10] He was chairman of the Mathematics Department at Bronx High School of Science at that time
[11] http://www.wfnmc.org/obitgreitzer.html
[12] The last issue was co-edited by Nura and David Ash, another winner of the USAMO, who also won the Canadian Mathematical Olympiad

paragraphs to compare the performances of Groups 1 and 2 and came out in favor of Group 1. Then she went on to say:

"*Just where might this 'compete-to-beat' sideplay end up? The 8 winners of the USAMO in accordance with competitions in other activities - academic or otherwise - have earned the right to provide the 6 members of a team to an IMO, and whether or not such a team beats others is not the important point. What is important is the early life experiences of team members mingling in friendly mathematical contact with young people of nations of the world with whom they are likely to mingle mathematically in future years.*"

Nura visited with us yet once more in Beaumont; the picture below shows her with our younger boys, both of whom became mechanical engineers.

I also had her on the program at ICME-6 in Budapest in 1988, where she gave a talk. That was the last time I saw her. The picture above shows some of the speakers in the Topic Area of Mathematical Competitions, for which I was the organizer at the Congress. In the front row from left to right Nura is the 2nd, while I am the 4th person.

She died 6 years later on the 26th of July 1994. She was 89 years old. Unfortunately, our mathematical community did not pay attention to her passing; hence the present necrologue, which I prefer to view as a celebration of her life and accomplishments.

Though she was relatively small in physical stature, Nura thought HUGE! Nobody else would have dreamed of asking the Department of State for their most lavish location as the site for the celebration of the winners of a mathematics competition. That huge hall, and the similarly sized and associated balcony with its magnificent view were only for the reception of kings and heads of state from other significant countries. And yet Nura managed to find the right persons in charge and convince them of the importance of our Olympiad.

The idea of dreaming of a national competition at the highest level in one of the biggest and most powerful nations should have also been too big for a second-class professor in a second-class state university even for a born New-Yorker from The City, and much more so for a person from Iowa, and especially for a single woman. Without a Ph.D. in a research-oriented university she was probably tolerated at best in spite of her accomplishments in the area of mathematics competitions. While she was given the Emerita title upon retirement, the website of the University's Mathematics Department fails to acknowledge her in its "In Memoriam" section.

Fighting the windmills may have been too ambitious for Don Quixote, but not for Nura. We should always remember her for that, learn from her example and follow it.

While this article is about the late Nura Turner and her accomplishments, I don't think that it takes anything away from its focus if I mention that I have been planning for a long time and still hope to write an article about the significant role of women in the history of mathematics in Hungary. Some of them were and others still are my friends, while many of them I know only from my own inquiries in the area. Just like Nura, they all deserve to be better known; that will be the purpose of my next article. Neither should it take away from the attention to Nura's story that I use this occasion to tell you about my own involvements, which were somewhat influenced by Nura's example. Forty years ago, I had my first opportunity to emulate the excellent example of my native country, Hungary's **KöMaL**, the world's first problems-driven mathematics journal[13] for high school students with problems in physics and informatics too. It was then that I accepted the challenge from my friend, the late Dave Logothetti, to make the 'Problems Section' of The **Mathematics Student (MS)** into a 'Competition Corner'. More precisely, as the editor of **MS**, he allowed me to create a year-round problem-solving competition, wherein the students were given about a month to solve 5 problems each of the months during which

[13]Its full name is Középiskolai Matematikai és Fizikai Lapok (meaning, 'High school mathematics and physics journal'. Earlier physics was included in a subtitle, now informatics has taken its place)

MS appeared[14]. Their solutions were evaluated, the evaluations were sent to the students and the scores awarded for them were accumulated. Those who gathered the most points during the year were given book prizes, as well as calculators, including programmable ones, thanks to Nura, who paved my way to the headquarters of Hewlett-Packard.

Unfortunately, the National Council of Teachers of Mathematics (NCTM) ceased the publication of the **MS**, and hence the 'Competition Corner' ended after only 3 years. Numerous attempts of mine to start a new publication failed, and soon I was chairing the MAA committee in charge of the 'American Invitational Mathematics Examination' (AIME), which consumed most of my time and energy for the next 6 years. Then I went on to start the USA Mathematical Talent Search (USAMTS), another attempt to transplant the spirit of **KöMaL** to America. There is an article about the USAMTS in this publication[15], as well as one about its international extension, the International Mathematical Talents Search [16](IMTS). The IMTS problems and solutions appeared in 2 volumes in Australia[17] in 2010 and 2011; they cover all 11 years of the IMTS, while the Appendix in Part 1 features the first 2 years of the USAMTS. Since the problems of the two programs were the same, the books cover the first 13 years of the USAMTS, which is presently in its 32nd year[18] under the auspices of the Art of Problem Solving (AoPS).

Due to the involvements mentioned above, my original hopes of publishing the materials of the 'Competition Corner' in a book had to be postponed time and again. However, now, 40 years later I am finally working on that book and in it on some sidebars concerning the former contestants and answering the question Nura addressed in her newsletters and books[19]: Where are they now?

Interestingly, there was an overlap between the Olympians of Nura and the participants of the Competition Corner not only in spirit, but in actual persons too. Most members of the latter group came via direct invitation sent to the non-senior members of the National Honor Role of the AHSME, i.e., those who scored 100 or more points (out of 150), while the ones invited to take the USAMO constituted only a subset thereof. Even among them, Nura's interests were limited to the winners of the USAMO, many of whom were in my program too.

During the years 1978-1981, about 400 students responded to my invitations by submitting solutions to some of the problems posed. Via teacher recommendations or otherwise their number rose to about 450. After nearly 40 years, thus far I have managed to renew my contact with about 100 of them. My plan is to feature as many of them as possible

[14] A total of 8 times in 1978-79, while only 6 times in 1979-1980 and in 1980-81

[15] George Berzsenyi, *The USA Mathematical Talent Search, Mathematics Competitions*, Vol. 9.2(1996), pp.40-46

[16] George Berzsenyi, *The International Mathematical Talent Search, Mathematics Competitions*, Vol. 10.1(1997), pp.74-78

[17] https://shop.amt.edu.au/collections/international-competitions-and-olympiads

[18] https://www.usamts.org/Problems/U_Problems.php

[19] In her excellent Mathematics and My Career, she featured seven essays by former contestants about their career choices and accomplishments, while in Who is Who in U.S.A. Mathematical Olympiad Participants, she followed the career paths of the winners of the USAMO between 1972 and 1986. Both were published by NCTM in 1971 and in 1987.

in the sidebars in response to the question: 'Where are they now?', while the main body of the text will show their excellent solutions to the problems posed. Two younger mathematicians, Gabriella Pinter and her husband, Istvan Lauko are my co-editors of the book, which should be ready for publication later this year. I would like to hope that Nura would have approved of our efforts in showcasing the accomplishments of former competitors.

Moreover, I would like to hope that Sam and Murray would happily welcome Nura's name to join theirs on the award given each year to the winner of the USAMO. All three of them have done a lot for the success of the USA Mathematical Olympiad, they all deserve the recognition.

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Team Competitions versus Individualism at Mathematics Houses in Iran

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Abstract
In Iran, like many other countries, individualism is known to be a problem especially among students. Mathematics Houses have put some emphasis on team-working activities. Most of the events in mathematics houses take place as participations of teams, e.g., teams of elementary school students, teams of high school students, teams of teachers, teams of university students, teams of university lecturers. As a result of this practice, many students who took part in IMH activities are known to be interested in cooperative research activities. The students of the house have also taken part in the International Mathematics Tournament of Towns competitions, A-lympiad competitions, Statistics competitions and problem solving day (competition for elementary school students) in teams. These competitions satisfy the other aim of the houses that is recognizing and fostering mathematically gifted students. In this article, we will try to show how the students enjoy their participation in these competitions in teams and how helpful they are for motivating the participants.

Introduction

Isfahan Mathematics House (IMH) ever since its establishment has believed that our society needs cooperation instead of individualism, and this attitude needs to be taught from school age. So the decision was made to do all the work at IMH in teams. All the work there, even the organization of IMH, has been done in an atmosphere of team work. All the training, all the competitions and all the projects in IMH are cooperative.

In this article we introduce the mathematics houses (the first one was IMH), and their mission and will discuss the usefulness and the necessity for team work, and finally we will show some of the activities of mathematics houses which are performed in teams, especially the competitions in which IMH or other houses in Iran are involved.

Mathematics Houses

Since 1999 in Iran, teams of teachers and university staff have established what are called Mathematics Houses. So far about 40 of them have been established. The first one was established in Isfahan and in 2000 (World Mathematical Year), mathematics houses began functioning in Yazd, Neishabour and Tabriz.

The Houses are meant to provide opportunities for students and teachers at all levels to experience team work by being involved in a deeper understanding of mathematics through the use of various media. Team competitions, e-competitions, using mathematics in the real world, studies on the history of mathematics, the connections between mathematics and other subjects such as art and science, general expository lectures, exhibitions, workshops, summer camps and annual festivals are some of the non-classic mathematical activities of these houses.

Students enjoy the atmosphere of cooperative working and exchanging information in these houses and this helps to enrich their mathematical knowledge and skills. Teachers and talented students may take part in all the activities or special programs of the houses, and in this way the houses nourish the higher achievers for better and higher achievements.

A Mathematics House is a playground and a center for providing good challenging foundations and its aim is to answer all the questions on the problems of the challenges in mathematics education.

It provides an up to date, state of the art experience for making necessary challenges for teachers and students. [1, 2]

Mission Statement of IMH:
A Mathematics House is a lively and creative research center with the following goals:

- Giving all members of the society the power of improving the quality of their personal and social life.
- Developing mathematical awareness in the society: using mathematical sciences in all aspects of life and work.
- Encouraging teamwork.
- Encouraging Interdisciplinary Research.
- Emphasis on incident learning.
- Teaching the skills for a better understanding of mathematical concepts. [3].

During the 13th International Congress of Mathematics Education (ICME-13), a Discussion Group on Mathematics Houses established an International Network of Mathematics Houses [4], which has proved to be a success for this new phenomenon in mathematics education. Those who benefit from the houses are mostly K-12 school students, teachers as well as university students, and even the general public for special exhibitions and general lectures. Every year more than 2000 school students take advantage of the different programs of each house.

Why Teamwork?

The first goal of mathematics houses is: "Giving all members of the society the power of improving the quality of their personal and social life." And this goal cannot be achieved without making the members of the house believe in teamwork. There are not many jobs available in the society for individual work. Every successful citizen should work with other members of the society regardless of the other person's background and ability. So each citizen should be prepared to work in teams and learn to respect other members of the society, while not considering him or herself superior to the others. Unfortunately, many competitions do not invite or prepare the winners to work in teams [5, 6].

As is explained in many studies [7], we believe in the following advantages of teamwork:

- Working in teams increases collaboration and allows brainstorming. As a result, more ideas are developed and productivity improves.
- Two or more people are always better than one for solving problems, finishing off difficult tasks and increasing creativity.
- Everyone is unique and has different skills, backgrounds and experiences. Therefore, others in a team can help you see things from a different angle.
- Teamwork encourages communication between team members. For this reason, relations between employees and the people who work together tend to be better and over time employees and those who are involved in teamwork learn to communicate better.
- Every person in a team has a different talent, and working in teams is useful for developing all talents.
- Some members of the team may think that their ideas are the best, but someone in the team, who does not seem to be talented thinks differently and who may not understand at first an idea advanced by a person on the team deemed to be more talented, and thus asks questions will be led to see a possible mistake and correct him or herself. (We have experienced this many times in the house and also in all the classes taught cooperatively there.)

There may also be some disadvantages of teamwork such as the following:

- In some teams, there may be members who sit back and let others do all the work. In this type of team conflicts may occur and this can affect the mood of others on the team.
- Working in a team requires many meetings and these meetings, if not managed well, can wander from the topic and decrease the efficiency of the team (although these days the meetings can be organized virtually and most of the time they keep focused).
- Making decisions can take longer while looking to form a consensus. Hence, delays occur.

Moreover, there are also some disadvantages of individual work such as:

- You are the sole responsible for the job. If you fail, it is your fault. There aren't any others to blame it for.

- You have to motivate yourself. There are no others to motivate you for getting things done.
- You can get bored working all by yourself. There isn't anyone to talk to, share ideas with or get help from.
- When you are working alone, if you get sick or need to take days off, the work will be delayed because there won't be anyone to continue it for you.

Finally we preferred teamwork over individual work due to the important fact that our society is exhausted from individualism; there are many people who are not able to work with others, even at home, because they think they are more talented than the others! [6], although the society needs teamwork more than individual actions.

A Report on Team Competitions at IMH

IMH, as the birth place of mathematics competitions in Iran (because the founders of school mathematics competition in Iran are the same as the founders of IMH) [8, 9, 10], believes in the effect of competitions in attracting talented students. For example, we have seen the real and wonderful fruits of mathematics competitions in Iran such as Maryam Mirzakhani [11], and would like to answer "Yes" to Prof. Man Keung Siu's question "Does society need IMO Medalists?" at his short talk at the IMO Forum on the theme "Mathematics in Society" in Hong Kong, which is a part of the program of the 57th IMO (International Mathematical Olympiad).[12]

We should not only think about mathematics, and not only about gifted students, but we should also think about the competitions and their effects on society and education. We should not continue our plan simply because we like problem solving practices! Rather we must learn to see the needs of the society at different times. One prescription does not work for all the patients or for the same patient at different times. Our society needs teamwork more than individualism, so we decided to organize team competitions to counteract the individualism among students.

International Mathematical Tournament of Towns

Starting in 1988, IMH entered the International Tournament of Towns (ITT) [20] with teams of students (although ITT is organised for recognising individual talents). The reasons for us not working with the Iranian Mathematics Olympiad are the following:

a) The first round of the Olympiad is run by a central committee and the teachers of mathematics and schools are not involved in the process of creating questions and organizing the first round. One of our aims has been to establish a process of creating good questions for the first round among mathematics teachers of different provinces (which is the case for choosing each country's Olympiad teams in the world). [5, 6, 9]

b) The questions in the first round are multiple choice questions which do not improve the ability of problem solving in students .

c) Let me refer to Professor Alexander Soifer's question in a report on the final round of the Third International Tournament Mathematics without Borders "Is 5 min/problem = math?" Our answer is "no" and we answered this question by entering the tournament of towns and ignoring other mathematics competitions.

The Mathematics A-lympiad

The Mathematics A-lympiad is a mathematical competition for teams of 3 or 4 students[21] organized by the Freudenthal institute of Utrecht University in the Netherlands.

As Professor Martin Kindt of the Freudenthal Institute in Netherlands describes in his notes [15], we were interested in mathematical modelling and in improving the ability of young students in modelling, as well as teaching the skills of using mathematics in real world. This led us to participate in this international competition, once again in teams; Iran has participated in the A-lympiad since 2009.

[20] The Tournament of Towns is an international mathematical Olympiad for school students of grades 8-11 (if 11 is the last grade). The peculiarity of the Tournament is dedication to a deep consideration of problems. It helps to develop qualities necessary for scientific research. See Andy Liu's column on the Tournament of Towns in this issue of *Mathematics Competitions*. The Tournament has been held annually since the spring 1980. The Tournament is founded and supported by the effort of enthusiastic mathematicians, university students and teachers. Many thanks to all of them! Participation in the Tournament is free for all school students. On the initiative of the President of the Tournament of Towns N.N. Konstantinov and with his participation, for organization of the Tournament and solution of related questions, the Center of Mathematical Olympiads "Tournament of Town" was established. [13]

[21] The teams work on an assignment - a very open-ended problem situation - in which mathematical problem solving and higher order thinking skills must be used to solve a real-world problem. The result of the assignment is a written report. The competition has two rounds: the qualifying preliminary round with about 1000 teams of students competing a day long at their own schools; and an international final in which 16 teams compete during a whole weekend in a central place in the Netherlands on a different assignment. [14]

Statistics Competitions

Our successful experience in organising mathematics competitions in Iran for attracting talents to mathematics [8, 11] and the need of society for statistically talented graduates made us eager to organise School Statistics competitions for teams of high school students in Iran in 2006, a report of it was presented at the WFNMC congress in Riga [16] .

Other activities of the house such as training teachers, elementary school students' playground with mathematics tools, high school trainings, university students' projects and other works at the house is also running in teams.

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[11] Maryam Mirzakhani; *The sweet fruit of student's math competitions*, Roshd, Magazine of Mathematical Education, 127 (2017), pp. 38-40, (in Farsi).

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A brief history of the South African Mathematics Olympiad

and what we find special about our Third Round for Junior Pupils

Thomas Hagspihl

Thomas Hagspihl is the chair of the South African Mathematics Olympiad

Abstract
The South African Mathematics Olympiad (SAMO) began just over 50 years ago in 1966. What makes the SAMO interesting is how it has changed over the years, not only as our country has changed politically and educationally, but also in an attempt to cater for a wider range of pupils.

In the first twenty five years of the SAMO's existence, apartheid was in full force in South Africa and the SAMO, in essence, catered for a small minority of mostly white, clever pupils. There were two rounds and the top one hundred pupils from the first round were invited to write the second round and then the medals were awarded. The first round was already quite tough and was aimed at the kids that did really well at Mathematics.

When democracy finally came to our troubled land, there was a strong feeling that the first round, at least, should be accessible to a far wider audience. What made this tricky – and we still struggle with this in our country today – is the huge disparity in the mathematical ability of high school pupils in our schools. Rural and township schools, through their lack of resources and good teachers, would typically not get much more than one or two of the twenty questions correct in the first round. In fact, even at many of the white schools, pupils would not do very well and so the sentiment about the SAMO was not very positive. An additional problem was that because teachers were not previously exposed to problem-solving and higher order thinking questions, they were reluctant to enter their pupils for the Olympiad. They might lose face with their pupils if they didn't know how to solve some of the problems. An attempt was made to overcome this challenge by providing detailed written solutions to all previous question papers but, to this day, there is still a distinct hesitancy amongst teachers to enter their pupils because many of them just cannot do the problems themselves – and are, in many cases, not even willing to try.

With all of this in mind, it was decided to introduce a system of three rounds. This gives the committee setting the papers, the opportunity to set a first round that is far more manageable for a much greater number of kids. The challenge is, of course, to find questions for the first round that are easy enough, and yet interesting with some real thinking required. There has always been an attempt in the SAMO to design and find questions that require as little of the formal school curriculum as possible – again, not always easy. This first round is marked in schools and every pupil who gets ten or more of the twenty questions correct goes through to the second round. We are aware of the possibility of cheating, but the second round is marked centrally by computer and the top one hundred pupils are invited to write the third round. Currently we have around 100 000 pupils that participate in the first round each year. Of these about 10 000 juniors (Grades 8 &9) and 8 500 seniors (Grades 10-12) get through to the second round. This change has had many positive spin-offs and we are pleased we made it.

Until recently both first and second round papers were multiple choice. To eliminate guessing, but more importantly to add a slightly different flavour, we introduced a second round paper that requires the pupils to calculate an actual numerical answer. This certainly has added a new dimension to our second round and since a computer marks these papers the SAMO committee is required to set questions in such a way that all the answers are integers from 1 to 999. The standard multiple choice techniques of eliminating obviously incorrect distractors, substitution and others – which the pupils get really good at and love using – are now no longer useful and a direct approach to solving the problems has to be used. We thus have multiple choice questions in Round 1, numerical answers for Round 2 and full written solutions for Round 3. It was another great change and we like it!

The last major change and the main topic of discussion for this article is the introduction of a separate Third Round for our junior pupils some 15 years ago. For most 13 and 14 year olds, the six questions in four hours at an IMO level was simply a waste of four hours. Many would get zero and hand in a blank script having studied the posters on the walls for most of the time. It was thus decided to set a separate question paper for Grade 8 and 9 pupils. 15 questions for 100 marks. The challenge the committee has set itself over the years is to produce a question paper that includes the following:

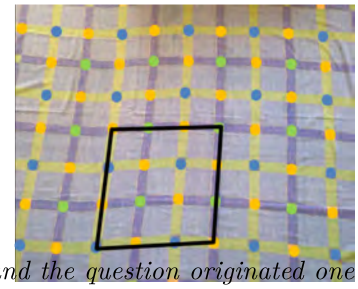
1. Some problems inspired by real life.
2. Some 'easy' proof-type problems.
3. We always try to include a games question that can be won using a strategy.
4. A classic age-old problem that everyone should get to know at some stage.
5. A puzzle.
6. Logic questions.
7. 'Wow, that is such cool mathematics' problems.
8. 'Surely that can't be – it's impossible' questions.
9. Easy, yet interesting problems with clever solutions.

We don't always succeed to include questions from all the categories but we certainly try. What follows are some examples of each category:

Problems inspired by real life

Example 1

The table cloth[a] in the picture consists of squares with differently coloured circles at their vertices. Indicated in black, you can see a 3×3 square containing 8 yellow circles. How many yellow circles are there on a $p \times p$ square?



[a] *This was an actual table cloth of one of the committee members and the question originated one Christmas morning when everyone else was still asleep*

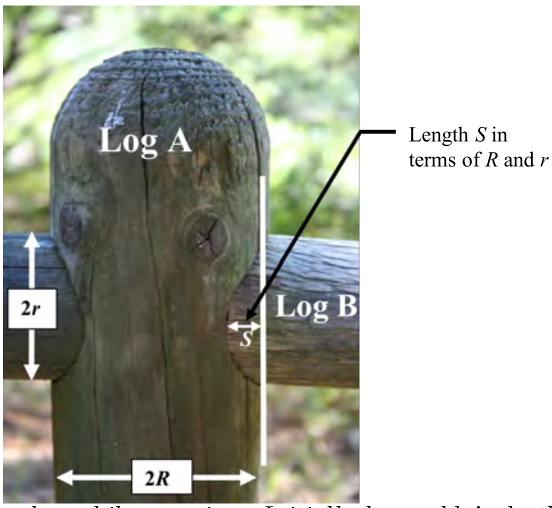
Example 2

The picture shows a gift[a] that you can buy in curio shops. It is a calendar which tells you the date and consists of two loose cubes which can be moved and rotated in any way. There must always be two numbers on display and in this case the date is 16 February. (Don't worry about the month which is displayed below the cubes) What numbers must be on the six faces of each of the cubes?

[a] *This calendar can be bought in many curio shops*

Example 3

In the diagram[a], the log A has radius R. A hole of radius r is drilled through the centre of log A at right angles to the axis. Another log B of radius r passes through the hole. Find the length S in terms of R and r.



[a] *Again, this photo was taken by a committee member while camping. Initially he couldn't do the problem and needed to be persuaded by his colleagues to put the problem into a paper. A real real-life question!*

Example 4

This is a real image of a Zimbabwean bank note.

The approximate thickness of one of these Zimbabwean bank notes is 0,1 mm. If we stack twenty trillion such notes on top of each other, how high would the pile be?

a. To the roof of your classroom?

b. As high as the Telkom tower in Pretoria?

c. As high as Table Mountain?

d. As high as Mount Everest?

e. As high as a 747 Jet flies from King Shaka International to Oliver Thambo International airport?

f. Higher than to the moon?

Explain your answer.

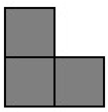
This is quite an interesting visual and with the current US national debt standing at around 23 trillion US$, this provides some interesting material for discussion!

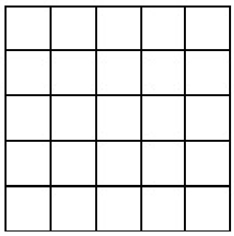
Easy proof-type problems

Example 1

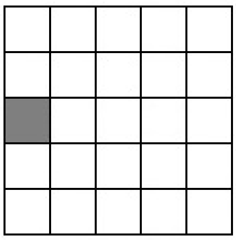
A 5-by-5 square consists of 25 1-by-1 small squares.

(a) Is it possible to tile this square with the non-overlapping L-shapes shown in the figure?

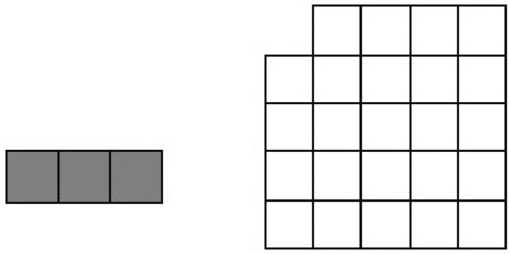




(b) If the shaded square is removed, is it possible to cover the rest of the square using 8 of the L- shapes shown above? (If it is possible, draw a solution. If it is not possible, prove it)

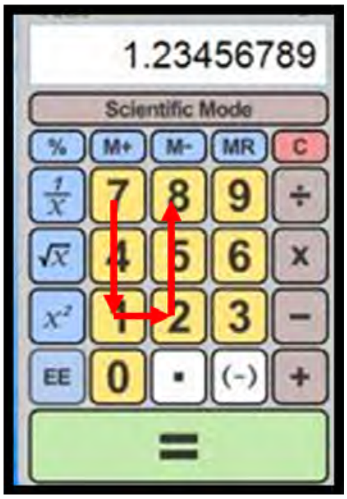


(c) If one corner square is removed, prove that it is not possible to cover the rest of the squares by eight 3-by-1 rectangles as shown in the figure.



Example 2

Did you know that, if you form a four digit number using any four non-zero digits on the corners of any rectangle on a calculator, the number will always be divisible by 11? In the example in the picture we have 7128. Look : which is an integer. 5236 also works: which is an integer.



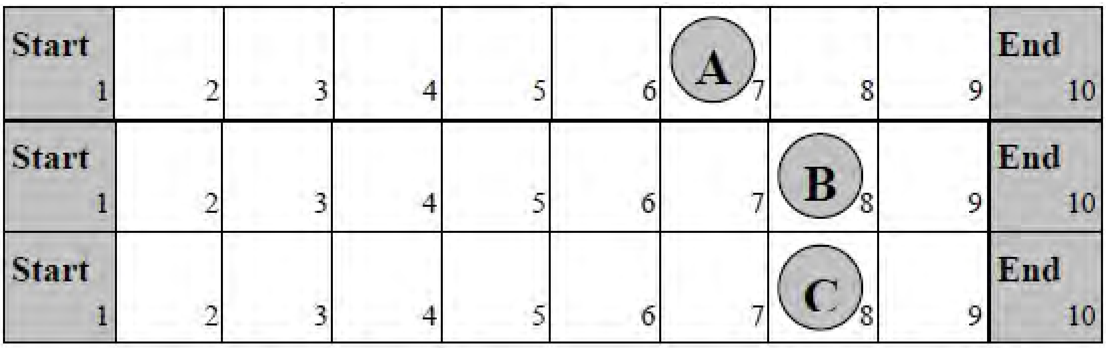
a) Prove it.

b) Prove that it even works if you rotate the calculator 90 degrees clockwise. (i.e., if you use numbers like 2365 or 4697)

Strategy game questions

Example 1

In the game “Move out”, two players take it in turns to move any one of the three counters on the board any number of squares to the right, beginning from their ‘start’ blocks, until all three counters are in the ‘end’ blocks. The last player to move a counter loses.



a) If it was your turn to play in the game above, which counter would you move and to which position, to guarantee you a win.

b) Explain your answer.

Example 2

Nick and John play the following game. They put 100 pebbles on the table. During any move, a player takes at least one and not more than eight pebbles. Nick makes the first move, then John makes his move, then Nick makes a move again and so on. The player who takes the last pebble is the winner of the game.

a) What strategy can you offer Nick to win the game?

b) Can you offer John, as the second player, such a strategy?

Example 3

A row of blocks is provided in your worksheet. Place the 2 Pula (*the currency in Botswana*) coins you have been given on any two squares on the grid. For example:

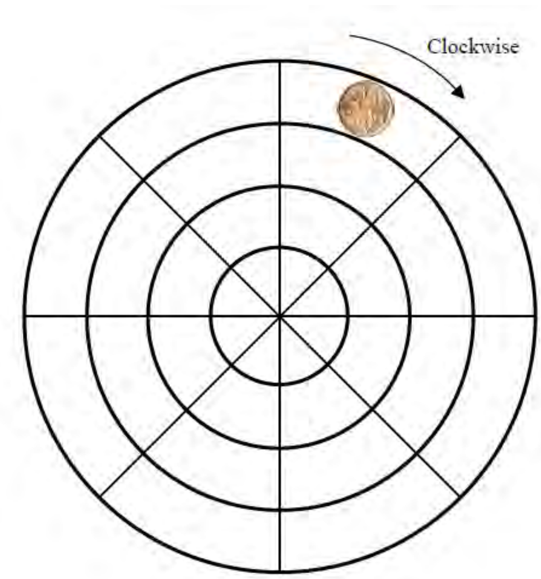
The Moving Coins game is an interesting game in which players take it in turns to move any one of the two coins any number of blocks to the right. You are not permitted to move back, or to jump over another coin. The first player who cannot move loses.

Play the game a couple of times to make sure you understand it.

Can the player who makes the first move always force a win? Explain.

Example 4

Zola and Ron play a game by alternately moving a single ten cent coin on a circular board. The game starts with the ten cent coin already on the board as shown. A player may move the coin either clockwise one position or one position toward the centre, but cannot move to a position that has been previously occupied. The last person who is able to move wins the game. If Zola starts, which player can play in a way that guarantees a win?
Explain this player's winning strategy.



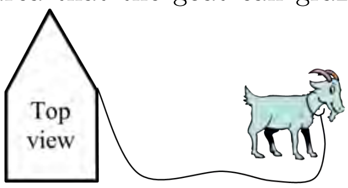
A classic well-known problem – some with a twist

Example 1

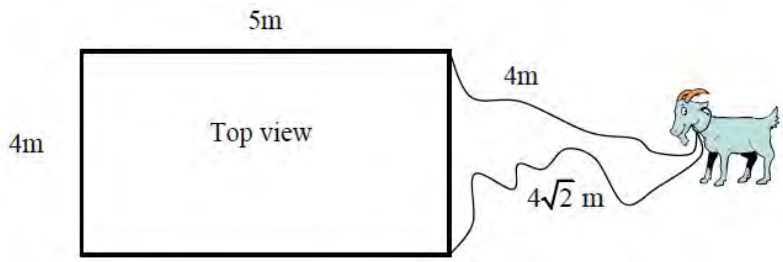
Potatoes are made up of 99% water and of 1% solid 'potato matter'. Vladimir bought 100 kg of potatoes and left them outside in the sun for a while. When he returned, he discovered that the potatoes had dehydrated and were now only made up of 98% water. How much did the potatoes now weigh? (This is also quite a 'wow' result)

Example 2

a) A goat is tethered to the corner of a shed which consists of a square and an equilateral triangle. The square has side length 2 m and the rope is 5 m long. What is the maximum area that the goat can graze outside of its shed? Give your answer in terms of π.



b) This time the goat is tethered to the corner of a rectangular shed 4m by 5m, but with two ropes of length $4\sqrt{2}$m and 4 m as shown. What is the area the goat can graze? Again, give your answer in terms of π.



Example 3

Place algebraic operations $+; -; \div; \times$ between the numbers 1 to 9, in that order, so that the total equals 100. You may also freely use brackets before or after any of the digits in the expression and numbers may be placed together, such as 123 and 67.

Two examples are given below:

i) $123 + 45 - 67 + 8 - 9 = 100$

ii) $1 + [(2 + 3) \times 4 \times 5] - [(6 - 7) \times (8 - 9)] = 100$

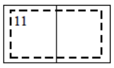
Four solutions will be awarded 2 marks each. Any other solution will get a bonus of 1 mark each to a maximum of 3 bonus marks.

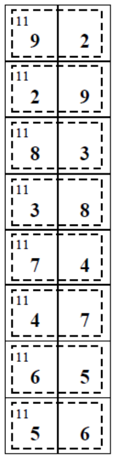
Puzzles

Example 1

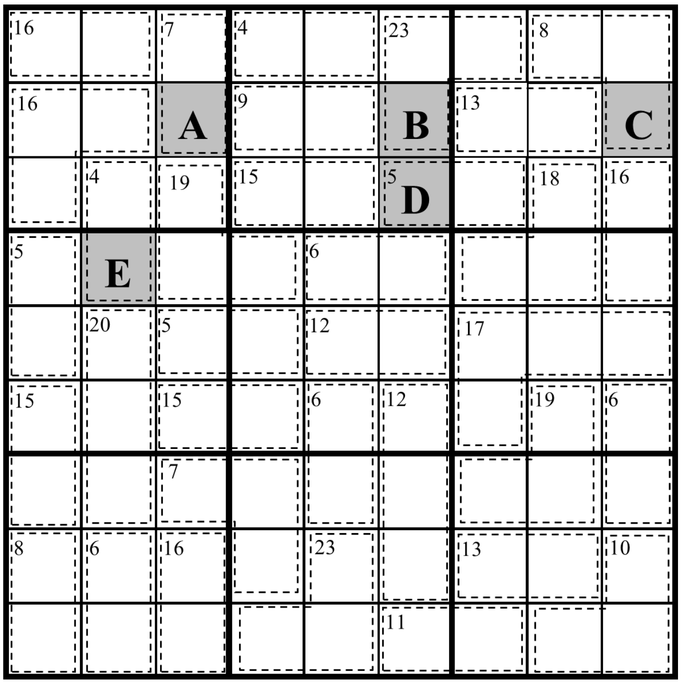
In a killer Sudoku, just like in a conventional Sudoku, the aim is to fill each row, each column and each 3-by-3 block with all the numbers from 1 to 9. Your clues in a Killer Sudoku are the caged numbers that represent the sum of the numbers within that cage.

Duplicate numbers cannot exist within a cage.

Keep in mind that  could be any of the following combinations:

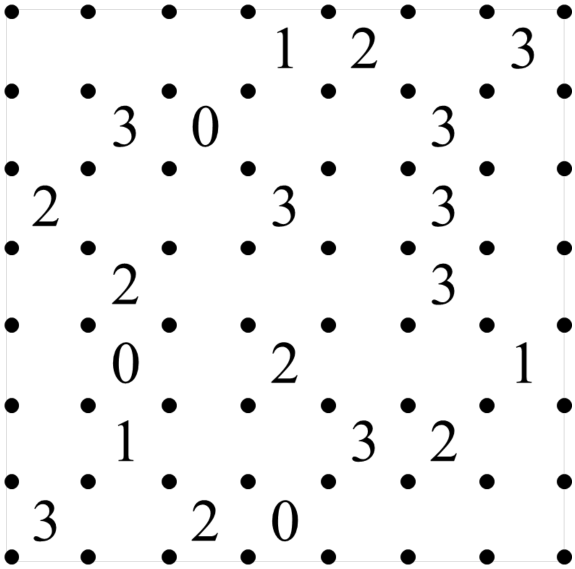


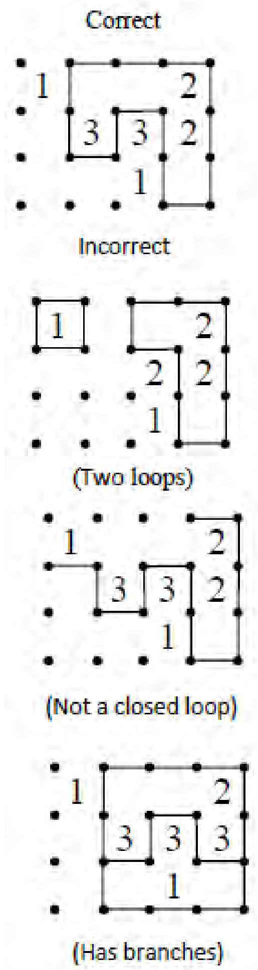
In the Sudoku below find the numbers represented by A, B, C, D and E.



Example 2

This puzzle is usually called ‘Slitherlink’. Connect adjacent dots with vertical or horizontal lines so that a single closed loop is formed with no crossings or branches. Each number indicates how many lines surround it, while empty cells may be surrounded by any number of lines or none. One correct and three incorrect examples are given.





Use the Answer Sheet to complete this question.

Logic questions

Example

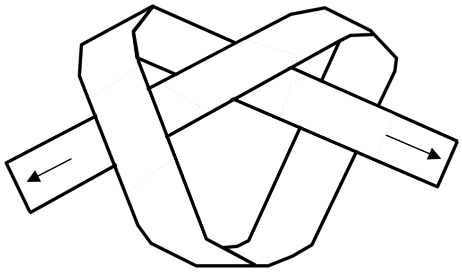
Prove that the alphanumeric, in which different letters represent different digits does not have a solution.

```
  T W E N T Y
+ T W E N T Y
-------------
C R I C K E T
```

“Wow – that is really cool mathematics” questions

Example 1

You have been given 2 strips of paper in your answer book. Tie a knot into one of the strips of paper (The second strip is just if you don’t get it right the first time)

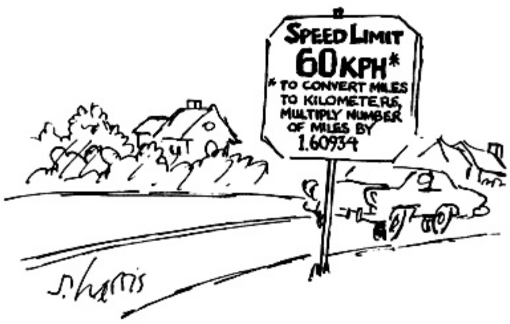


(a) What shape do you get? Cool hey!!

(b) You will see that in some places your shape is 2 layers, 3 layers and 4 layers thick. Find the shape that is 4 layers thick.

Example 2

The Fibonacci sequence is given by $1; 1; 2; 3; 5; 8; 13; 21; \ldots,$ where the next number is generated by summing the previous two. Fibonacci numbers were made famous by the rabbit problem, because it explained rabbit breeding. It is less well-known that one can also use Fibonacci numbers to convert miles to kilometers. To do so, one must realise that every positive integer can be uniquely expressed as the sum of different, non-consecutive Fibonacci numbers. To convert integer miles into kilometres, miles are expressed as the unique sum of non-consecutive Fibonacci numbers, then each Fibonacci number is changed to the next Fibonacci number.
The new sum approximately gives the kilometres. For example, 50 miles $= 34 + 13 + 3$ miles, where each number on the right hand side is a Fibonacci number. Using the conversion above, the right-hand side becomes $55 + 21 + 5 = 81$ km. Now use this method to convert 120 miles into kilometres. Show your work.
(You might want to check your answer by using the conversion on the cartoon.)



“Surely that can’t be” problems

Example 1

On a distant planet, railway tracks are built using one solid railway bar. A railway is built between two towns 20 km apart on a big flat section of the planet. Unfortunately the bar was made one metre too long and the constructor decided to lift it in the middle to try to make the ends fit.

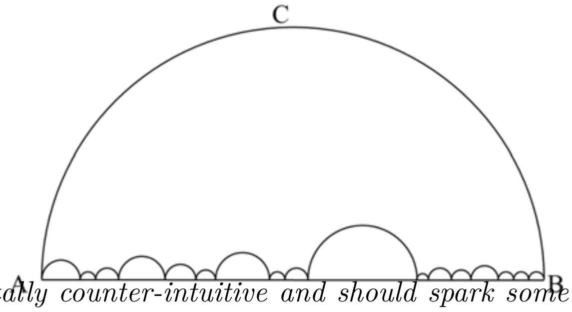
Approximately how high does he have to lift it in the middle? Is it 1 cm, 10 cm, 1 m, 10 m, 100 m or 1 km?

a) Guess one of the above, without doing any calculations.

b) Calculate the answer and comment on how it compares with your guess.

Example 2

When you travel from A to B, you can either travel along the big semi-circle (i.e., via C) or you can travel along all the smaller semi-circles.
Which is the shorter route and why?

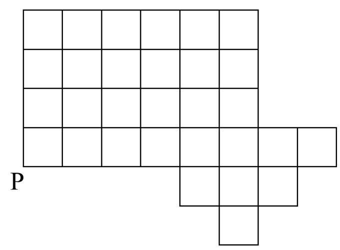


The solutions to both these problems are totally counter-intuitive and should spark some interesting discussions amongst the pupils.

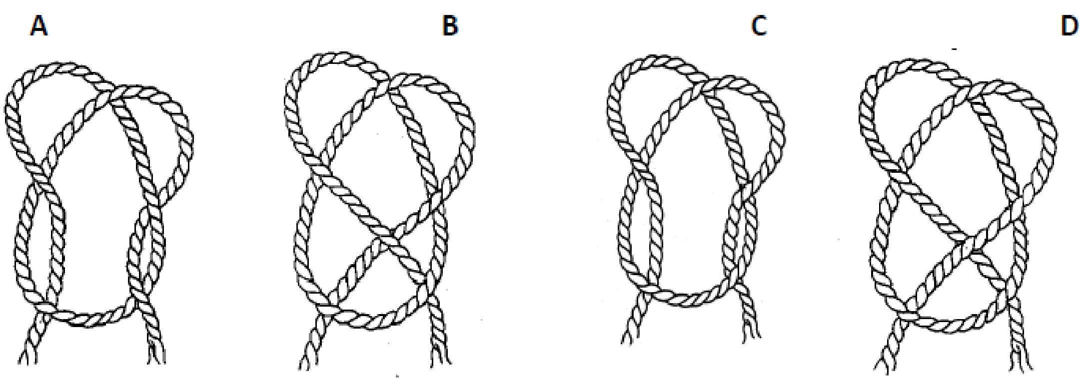
Easy, interesting problems with clever solutions

Example 1

The figure is made up of squares. Draw a straight line through P to divide the shape into two equal parts and explain why your line in fact works.



Example 2



Grasp the two loose ends of each rope firmly in your mind. Then imagine yourself pulling them until you have a straight piece of rope – either with a knot or without one. Which of these four ropes will give you a knot?

Example 3

a) Calculate: $\sqrt{4 \times 3 \times 2 \times 1 + 1}$.

b) Determine: $\sqrt{51 \times 50 \times 49 \times 48 + 1}$.

c) Find and prove the general formula for the square root of the product of four consecutive integers plus 1.

Something to take home

Finally, we always like to give each of the kids something to take away with them.

Two examples. . .

A Botswana 1 Pula coin is not circular, but nevertheless has a constant diameter. Two Pula coins and 2 rulers are given to you. Place the Pula coins on the space provided with the rulers besides them (see worksheet). Move the rulers back and forth so that the Pula coins roll between them. Wow – no bumps?!![a] If this constant diameter of the coin is Q, what is the perimeter?

[a] *This is an amazing result and they'll keep the ruler and coin to show everyone at home.*

A certain type of ring has an outer diameter of 58 mm and an inner diameter of 40 mm and a thickness of 1 mm. If one stacks enough rings on top of each other, it is possible to stand another ring vertically on top of the pile in such a way that the ring doesn't touch the ground. What is the minimum number of rings you need to stack on top of each other so that the vertical ring just doesn't touch the ground? (You have been given some rings to help you with this question. They do not have the same dimensions as in the question but you can build the model with them if you need to)

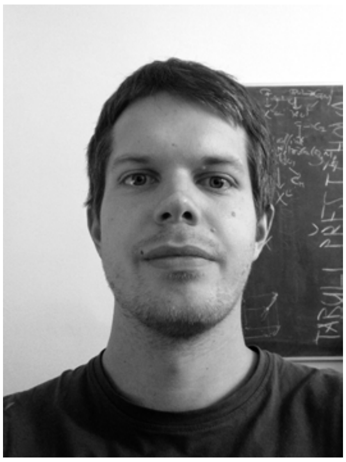
On occasions we have received criticism from various people, particularly from the hard core mathematicians, that a number of our questions aren't 'mathematical'. We do understand this criticism and are fully aware that many questions are not strictly pure mathematics – whatever that may mean. However, for us, the overriding principle is that the pupils who write our paper must be challenged, learn something new and interesting and come away with at least one 'Aha' moment. They must have enjoyed the experience with lots to talk about long after the paper has been completed. We have realised that setting good Olympiad papers is a form of Art and producing a masterpiece is not entirely simple. But oh, what fun it is!

NáBOJ – A SOMEHOW DIFFERENT COMPETITION

Erich Fuchs, Bettina Kreuzer,
Alexander Slávik, and Martina Vávačková

Dr. Erich Fuchs received his diploma in computer science from the University of the Armed Forces in Munich, Germany in 1984 and his Ph.D. degree in computer science from the University of Passau, Germany in 1999. Currently, he is the managing director of the Institute for Software Systems in Technical Applications at the University of Passau. Furthermore, he established a math circle for talented high school students at the University of Passau and is involved in the organization of national mathematics competitions and the international Náboj competition.

Bettina Kreuzer received her M.Sc. at the Queen's University in Kingston, ON, Canada in 1991 and her diploma in mathematics at the University of Regensburg, Germany in 1994. After moving to Passau, her children attended the math circle at the University of Passau, and in 2009 she herself became involved there. She loves to teach math circle courses and to organize various mathematics events and competitions for high school students—among them the Náboj competition.



Alexander Slávik (mostly known as "Olin" among Náboj organizers) competed in Náboj twice before assuming the role of an organizer in 2012. Since then, he has supervised the contest problem selection several times, prepared hundreds of PDF files, and helped with developing the online system for Náboj. Unfortunately, he is the only person who understands the whole thing. Currently, Olin is finishing his PhD studies of mathematics at Charles University, Prague, where his research mainly focuses on commutative algebra, algebraic geometry, and homological algebra.

Martina Vaváčková attended Náboj as a competitor in 2008, 2009 and 2010. She became a member of the organizing team in 2011, focusing primarily on public relations, budget management, integration of new competition sites, and proofreading. In the period of 2015 to 2017, she was the chief organizer of Náboj in Prague, but now she works mostly on international expansion. Martina studied mathematics at the Charles University and completed her master's degree in general and combinatorial algebra in 2018.

Abstract
In this article we present the Náboj competition which is an international dynamic math competition designed for teams of five high school students who represent their school. There are various competition sites in several countries where the teams meet at the same time and compete against each other. Within 120 minutes, they try to solve as many given problems as possible. Since the results are recorded and stored online in a central database, students can compare their performance in the national as well as the international ranking via internet during the competition in real time.

Introduction

The Náboj competition has been popular ever since it started in 1998, and its expansion to various countries in Europe has made it even more popular. Last year (2019), around 5000 high school students took part in competition sites in 17 cities and ten countries. The original idea was to come up with a team competition as a counterpart to the various individual math competitions.

As the name Náboj itself suggests in the Slovak and Czech language (the literal meaning of the word is *bullet* or *projectile*), this competition is very energetic, dynamic and full of drive and vitality, because in addition to solving math problems, contestants have to physically move, sometimes even run in order to promote the team.

There are two categories: Junior and Senior. Junior teams have members who are in grades not in the last two years of secondary education, and a team belongs to the Senior category as soon as one member is in the last two grades. Because Junior teams get ten easier problems to solve before they continue with the Senior problems of increasing difficulty, this division is a way to take the differing average levels of knowledge of the participants into account.

Many high school students are astounded about the fact that Náboj problems are different from math problems they encounter at school. Náboj problems start at an easy level, for which common sense, cleverness and creativity suffice to find a solution. These problems are followed by tasks of medium difficulty, which require skilled application of techniques learned at school and other math activities. Finally, there are problems that come close to approaching the level of established international competitions in order to offer something for the highest level students.

As is so often the case, problems are taken from the areas of algebra, combinatorics, geometry, and number theory. The answers to the problems usually consist of one number or a few numerical values that can easily be checked. Since only the answer has to be written on the problem sheet for checking the solution, the problem posers make sure that the correct answer cannot be guessed. Some elite teams sometimes manage to receive all the prepared problems during the competition, but solving all the problems is extremely rare.

All problems are stated without any hint of a possible result and have to be solved within 120 minutes using pencil, paper, ruler, and compass only—text books and electronic devices are not allowed. Only the organizers have access to electronic devices, as they have to use computers, scanners and internet connection in order to send the correct results given by the teams to the server and receive the live ranking lists from the central server in Prague.

Benefits of this Competition

Since the level of difficulty of the problems rises gradually, the competition is well accepted both in the community of regular high school students with an interest in mathematics and among students already involved in national math competitions.

In fact, this competition has positive effects on both of these groups of students. On one hand, Náboj gets many students of the by far larger first group involved in and hooked to solving math problems. In contrast to the national Olympiads, it is not required to give a complete proof during the competition, which eliminates an aspect some would consider a barrier to participation. On the other hand, apart from a few special schools, the members of the second group are sparsely spread, and only the best of them can make it to the final round of their national Olympiads. The Náboj team competition is therefore an excellent venue for students who stand in the second row to show their abilities, to contribute to their team's advancement and to promote solving of math problems among their fellow students. Since it is beneficial for a team to both use the strength of each single team member and the collaboration within the team, students of the two groups will definitively get closer and this helps to broaden the basis of high school students who will eventually get seriously involved in solving math problems.

After the competition, all problems and answers including full proofs are available on the archive of the Náboj website. Everybody can have a look at the problems, solve them, and check the result as well as their proofs. Actually, many Náboj contestants do exactly that as soon as the competition is over.

Preparations in Advance

Since 1998, the Náboj team competition has been held annually. One year before the competition, the local organizers of the Náboj competition sites in the various participating cities and countries agree on the date for the next competition, which takes place sometime in the spring. National holidays are taken into account, as are the dates of other math and science competitions.

Usually at the start of the university winter term, it is time for collecting and proposing problems. This is done on an open source CMS, using English as common language. Contributors of original or modified existing math problems are mainly math students and former math students who continue to work at the university. Most of them have either participated in past Náboj competitions or have some experience in national or international math competitions, while others are involved in teaching maths in high schools or math circles.

After a set of about 55 math problems is chosen, this set of problems is tested by some groups of university students and professors who are close to Náboj problem posers. The recommendations of the testers are then implemented, and the English version is finalized. Every participating country has its own group of people who then translate the selected problems into their own language.

How to Enter the Competition

Five weeks before the competition date, teams can register at the Náboj website to take part in the competition. Due to the limited capacity of rooms at a local competition site, only a certain number of teams can be admitted. A single high school can generally send one Junior team and one Senior team, but some exceptions may apply. For further information see the section about the rules on the website.

Competition Day

The competition takes place on a fixed date and starts at exactly the same time at each of the competition sites in different cities in various countries. It ends after exactly 120 minutes. At the start of the competition, participating teams receive an envelope containing the first 6 problems. The difference between the Junior and the Senior section is that Juniors start with problems numbered 1–6, while Seniors start with problems numbered 11–16. The members of a team work on these problems together, i.e., they are allowed to communicate within their team. As soon as the team feels that it has solved a problem, their answer is written on the problem sheet and a member of the team goes to a control post to check the correctness of the answer. An incorrect solution is crossed out and the problem goes back to the team for further examination. A correct solution is marked by a stamp and then handed over to the scanning station, where the team is given the credit for the solved problem via its bar code label. At this time, the team is also given the next problem.

This goes on until the time limit of 120 minutes is reached. At the very end, the contestants are allowed to line up one last time to check their results.

Due to the individual bar codes printed on the problems, the ranking of the teams can be shown immediately during the competition via live stream. This live stream is frozen during the last 15 minutes of the competition in order to preserve the suspense concerning the ultimate winners.

One great advantage of this competition is that almost all organizational work can be done in advance, the winning teams are determined by the supporting computer system, and marking of solutions after the competition is not necessary.

The History of Náboj

It seems that lively competitions of this type had already been around for decades when this specific competition was conceived. A similar competition was organized regularly at the camps of the Slovak Correspondence Seminars, but in 1997 a student named Richard Kollár came up with the idea of organizing this for the general public as a complement to the mostly individual math competitions. A "test" event took place that year for a small number of invited schools from Bratislava and since everyone liked the idea, an all-Slovak competition was then held in Bratislava in 1998.

The expansion of Náboj to Prague is due to a Slovak student named Marek Tesař. He was a big fan of the Bratislava competition, and having moved to Prague to study mathematics, he started promoting the idea among the organizers of the Prague Correspondence Seminar. Luckily for him, the Slovak organizers had already considered holding the competition at more than one venue at the same time and had prepared all the software needed. Thanks to this coincidence, Prague was able to organize its first Náboj in 2005. In that year, the division into two categories was established, albeit only in Bratislava. The Junior category did not become standard until much later, namely in 2011. Until 2013, the competition was held in two Slovak cities (Bratislava, Košice) and in two Czech cities (Prague, Opava).

The idea to spread Náboj beyond the borders of former Czechoslovakia had already existed for several years before the expansion actually took place. It was then mostly due to personal connections of Náboj organizers to people doing some additional math training with talented high school students that the expansion of the competition was possible.

During her time in high school, Martina Vávačková took part in the Náboj competition as a contestant, and at the same time she was part of a group of Czech pupils who were invited to participate in the German Math Olympiad organized by Erich Fuchs in Passau. Some years later, when she was a student at the Charles University in Prague, she became involved in organizing the Náboj competition and she then reactivated her connection to Passau. In 2013, Erich went to Prague with nine teams of high school students from the region of Passau to run a small German Náboj as a test. Due to the overwhelming acceptance on the part of the pupils, Passau became a competition site in 2014. And again thanks to personal connections, it then spread to Linz (Austria) the very next year.

Also in 2015, Náboj expanded to Budapest (Hungary) and Kraków (Poland). In the following year, more cities in Hungary and Poland became Náboj sites.

The competition in Edinburgh started in 2017, when some Slovak organizers of "Náboj Junior" (a primary school competition) moved there, and it is organized mostly by local Slovaks. The next addition was Constanţa in Romania in 2018—they joined in after simply discovering the Náboj website and asking whether they could participate.

The expansion of Náboj last year (2019) was mainly due to the fact that Náboj organizers tend to take along the Náboj competition when moving into another country. The new site in Cambridge (UK) was organized by Czech students. Novosibirsk (Russia) took part for the first time—the head organizer there had already been involved with the competition during his stay in Linz (Austria). The latest venue in Zurich (Switzerland) actually was initiated in two ways: again Marek Tesař moving, this time to Zurich, and the presentation of the Náboj competition at the 8th International WFNMC Congress in Semriach (Austria) in 2018.

Some Náboj Problems

Typical Aspects of Náboj Problems

We would like to present some examples of problems used in the competition. For this purpose, we have chosen four problems which reflect typical aspects of Náboj to be considered during problem selection. Contestants do not have to deliver a complete mathematical proof, but only one or few numerical values as a solution. In contrast to multiple choice competitions, like Kangaroo of Mathematics, they do not even know possible values coming into consideration. But still problem proposers have to be careful that the correct answer cannot be guessed easily. On the other hand, the problems should allow a differentiation between strong and less experienced teams, e.g. through various approaches to the solution requiring discriminative efforts.

Algebra: Náboj 2015, problem 51

Problem. Find all real m such that the roots of the equation

$$x^3 - 15\sqrt{2}x^2 + mx - 195\sqrt{2} = 0$$

are side lengths of a right-angled triangle.

Result. 281/2

Solution. Let a, b, c be the roots of the given equation, which are also side lengths of a right-angled triangle. Assume w.l.o.g. that $0 < a, b < c$, hence by the Pythagorean theorem, the equation $a^2 + b^2 = c^2$ holds. By Vieta's formulas (or working out $(x-a)(x-b)(x-c)$ and then comparing coefficients), we get

$$15\sqrt{2} = a + b + c, \quad m = ab + ac + bc, \quad 195\sqrt{2} = abc.$$

Squaring $15\sqrt{2} - c = a + b$ leads to $450 - 30\sqrt{2}c = 2ab$. After multiplying by c and substituting $abc = 195\sqrt{2}$, we get the quadratic equation

$$\sqrt{2}c^2 - 15c + 13\sqrt{2} = 0,$$

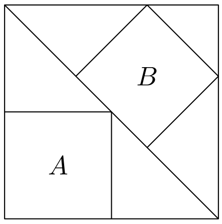
having roots $c_1 = \sqrt{2}$ and $c_2 = 13\sqrt{2}/2$. Since the constraints $0 < a, b < c$ and $abc = 195\sqrt{2}$ allow only $c = 13\sqrt{2}/2$, the desired number m can be calculated via

$$m = ab + ac + bc = \frac{1}{2} \cdot ((a+b+c)^2 - 2c^2) = \frac{1}{2} \cdot 450 - c^2 = 281/2.$$

The original statement during the problem selection process was formulated with the equation $x^3 - 12x^2 + mx - 60 = 0$ leading to $a = 3$, $b = 4$ (or vice versa) and $c = 5$ giving the result $m = ab + ac + bc = 47$ by analogous calculations. However, a team arriving at problem number 51 knows that this kind of exercise can be tackled using Vieta's formulas and from $a + b + c = 12$ and $abc = 60$ one can easily derive the well-known side lengths $3, 4, 5$ of a right-angled triangle and just guess the solution. Since almost at the end of the competition the teams should make an effort to get the solution, we changed the constants in the problem statement accordingly.

Geometry: Náboj 2019, problem 9

Problem. Two squares are inside a larger square as in the picture. Find the area of the square A if the area of the square B is 48.



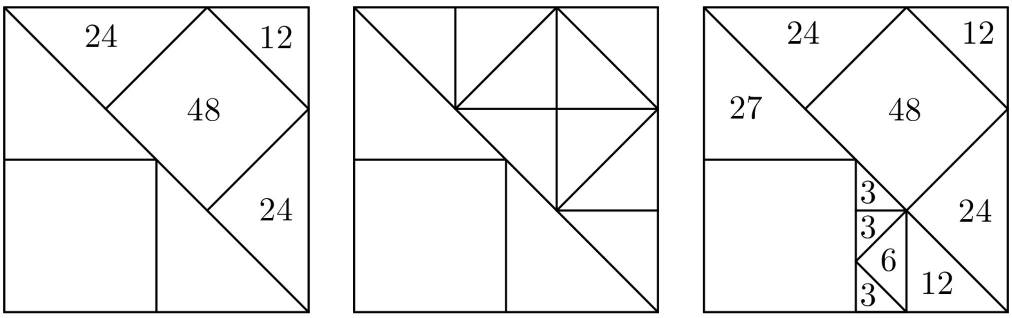
Result. 54

Solution. Since the triangles adjacent to the sides of square B are isosceles, the side of B lying on the diagonal is precisely the middle third of the diagonal. Therefore, if s is the side length of the large square, the side length of B is $\frac{1}{3} \cdot \sqrt{2} \cdot s$ and the one of A is $\frac{1}{2} \cdot s$. Therefore, the area ratio of the inscribed squares is

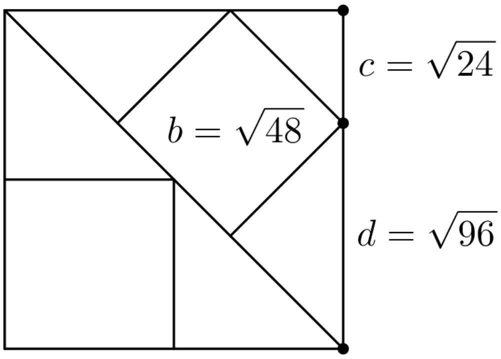
$$\frac{s^2}{4} : \frac{2 \cdot s^2}{9} = \frac{9}{8}$$

so the area of square A is $48 \cdot \frac{9}{8} = 54$.

Above calculation was the official solution. However, several clever teams used one of the following tilings to see that the area of half of the large square is $48 + 24 + 24 + 12 = 108 = 9 \cdot \frac{48}{4}$ leading immediately to the solution with almost no computations. One team continuously took half of the size of an isosceles right-angled triangle ending up in triangles of area 3.



Most of the teams computed the side length b of square B and from that either the side length or the diagonal of the large square leading to formulas like $\left[\frac{1}{2}(\sqrt{24}+\sqrt{96})\right]^2 = 54$ or $\left(\frac{3\cdot\sqrt{48}}{2\cdot\sqrt{2}}\right)^2 = 54$.



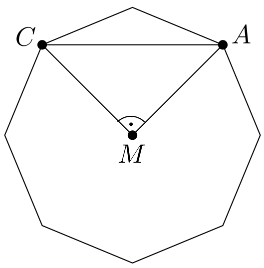
This is a good example where experienced teams can save valuable time even in this early stage of the competition compared to weaker teams which lose time due to sometimes erroneous computations using more complicated formulas.

Geometry: Náboj 2019, problem 23

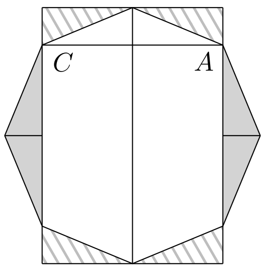
Problem. Let $ABCDEFGH$ be a regular octagon with $AC = 7\sqrt{2}$. Determine its area.

Result. $98\sqrt{2}$

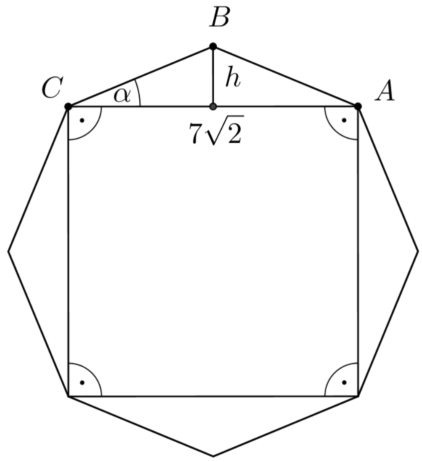
Solution. Let M be the center of the circumcircle of the given octagon. Since $\angle AMC = \frac{2}{8}\cdot 360° = 90°$, the radius of the circumcircle has to be 7 and the diameter is 14.



Based on the area rearrangement shown in the picture, we get the area by multiplying AC with the diameter. Therefore the area sought is $14 \cdot 7\sqrt{2} = 98\sqrt{2}$.



Only a few teams saw this area rearrangement and solved the problem very quickly, even below one minute. Most of the teams chose to inscribe a square of area $(7\sqrt{2})^2 = 98$ into the octagon and to compute the area of ABC.



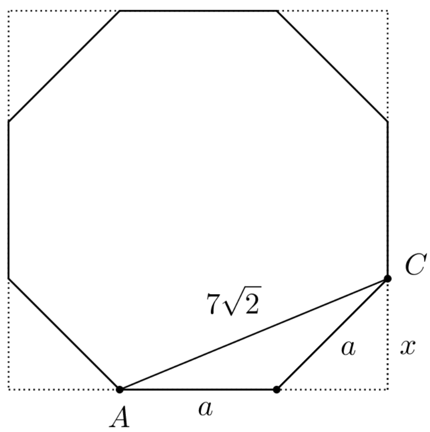
Since the interior angles of the regular octagon have size 135°, we get $\alpha = 22{,}5° = \frac{\pi}{8}$ and therefore $h = \frac{1}{2} \cdot 7\sqrt{2} \cdot \tan(\frac{\pi}{8})$. As a consequence, the area of triangle ABC is $A_{ABC} = \frac{1}{4} \cdot (7\sqrt{2})^2 \cdot \tan(\frac{\pi}{8})$ and the area sought is

$$98 + 4 \cdot A_{ABC} = 98 \cdot \left(1 + \tan\left(\frac{\pi}{8}\right)\right) = 98\sqrt{2}$$

due to $\tan(\frac{\pi}{8}) = \sqrt{2} - 1$.

Although these teams saw that the inner square is surrounded by eight congruent triangles, they did not recognize the rearrangement and had to compute a little bit more.

Another approach the teams chose was to circumscribe the octagon with a square and to find the side length a of the regular octagon using the Pythagorean theorem.



Since x is the side length of a square with diagonal a we have $x = \frac{a}{\sqrt{2}}$. Substituting this in the equation

$$x^2 + (a + x)^2 = (7\sqrt{2})^2,$$

we get $a^2 = \frac{98}{2+\sqrt{2}}$ and the area of the octagon from

$$(a+2x)^2 - 2x^2 = a^2(2+2\sqrt{2}) = 98\sqrt{2}$$

by straightforward computation. A huge majority of the teams solved this problem. However, the time needed to find the solution showed a large variance as desired.

Algebra: Náboj 2015, problem 44

This problem allows various algebraic approaches using different systems of equations and even a geometric access via the theorem of Menelaos is possible. However, all these approaches require a certain effort as intended for a problem in the forties.

Problem. Four people were moving along a road, each of them at some constant speed. The first one was driving a car, the second one was riding a motorcycle, the third one was riding a Vespa scooter, and the fourth one a bicycle. The car driver met the Vespa at 12 noon, the bicyclist at 2 p.m., and the motorcyclist at 4 p.m. The motorcyclist met the Vespa at 5 p.m., and the bicyclist at 6 p.m. At what time did the bicyclist meet the Vespa scooter?

Result. 3:20 p.m.

Solution. Since the time in question does not depend on the chosen reference frame, we may assume that the car is not moving at all. Under this assumption, the motorcycle needed one hour from where it met the car to its meeting point with the Vespa, whereas the Vespa needed five hours for the same distance, thus the motorcycle was five times faster. Similarly, one may deduce that the motorcyclist was twice as fast as the bicyclist, hence the ratio of the speeds of the Vespa and the bicycle was 2 : 5.

If the Vespa needed t hours to get from the car to its meeting point with the bicyclist, then the bicyclist needed $t-2$ hours. The ratio of these required times equals the inverse of the ratio of the speeds, so

$$\frac{t-2}{t} = \frac{2}{5},$$

or $t = 10/3$. Finally, using the fact that the Vespa met the car at 12 noon, we infer that it met the bicyclist at 3:20 p.m.

Second Solution. According to the text, the car and the scooter move in one direction and both motorcycle and bicycle are oncoming traffic. Denote the distances covered by the car, the scooter, the motorcycle, and the bicycle in one hour by d_c, d_s, d_m, and d_b, respectively. Since the car and the scooter are in the same location at noon, the distance from this meeting point to the motorcycle is the distance $4d_c + 4d_m$ between the car and the motorcycle on one hand, and the distance $5d_s + 5d_m$ between the scooter and the motorcycle on the other hand. This leads to the equation

$$4d_c = 5d_s + d_m.$$

By analogy for the meeting point of the car and the bicycle at 2 p.m., we obtain the equation

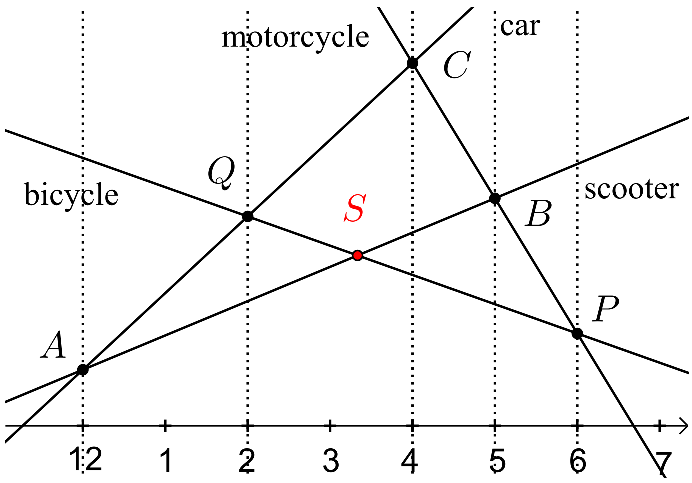
$$4d_b + 2d_c = 2d_m.$$

Eliminating d_m yields

$$3d_c = 5d_s + 2d_b. \qquad (*)$$

At noon, the distance between the car and the bicycle is $2d_c + 2d_b$. But this is also equal to the distance $t \cdot d_s + t \cdot d_b$ between the scooter and the bicycle, where t denotes the elapsed time between noon and their encounter. In this equation, substitute d_c from $(*)$ to finally get $t = 10/3$. Therefore, the scooter met the bicycle at 3:20 p.m.

Third Solution (Geometric Solution). In a time-position diagram, all involved vehicles move on straight lines, since they move at some constant speed. These four lines have six intersection points, for which at five of them the time coordinate is given. The positions of the meetings have to be at the intersection of corresponding lines in the diagram. Thus, we can determine the sixth intersection point, and hence its time coordinate, by using the theorem of Menelaos as follows:



Let A be the position where the car caught up with the scooter, Q the position where it encountered the bicycle and C where it met the motorcycle. Further, let P be the position where the motorcycle caught up with the bicycle. So, the car moves along the line AC, the scooter along AB, the bicycle along QP and the motorcycle along CP. We are looking for the point S, where the scooter and the bicycle meet.

Applying the theorem of Menelaos to the triangle ABC and the line PQ yields

$$\frac{\overline{AS}}{\overline{SB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CQ}}{\overline{QA}} = -1.$$

We have $\frac{\overline{BP}}{\overline{PC}} = \frac{-1}{2}$ and $\frac{\overline{CQ}}{\overline{QA}} = 1$, which implies $\overline{AS} = 2 \cdot \overline{SB}$. If x_s denotes the time coordinate of S, we then obtain $\frac{2}{3} = \frac{\overline{AS}}{\overline{AB}} = \frac{x_s}{5}$ and hence $x_s = \frac{2}{3} \cdot 5 = 3\frac{1}{3}$.

This means the scooter met the bicycle at 3:20 p.m.

Problems in Number Theory of Increasing Difficulty

In number theory, we present a series of problems ranging in difficulty from very easy to hard, in order to demonstrate the typical increase in the levels of difficulty during the course of the competition.

Náboj number theory problems start with school knowledge about prime numbers, divisibility rules including the sum of digits, and prime factorization. At medium level it is advantageous to know congruences. The lack of this tool means having to put more effort on getting the solution. More experienced teams save time which they need for the harder problems like problem 49 of Náboj 2019. At this level, contestants need deeper knowledge in divisibility, an observant eye and arithmetic skills like the technique of substitution or the binomial expansion to compute the solution.

Náboj 2012, problem 3

Problem. A five digit number $a679b$ is divisible by 72. Find the value of $a \cdot b$.

Result. $3 \cdot 2 = 6$

Solution. Since $a679b$ is divisible by 8, we get $b = 2$. Since it is divisible by 9, the condition $9 \mid a + 6 + 7 + 9 + 2$ yields $a = 3$.

Náboj 2011, problem 13

Problem. Find the smallest positive integer which is divisible by 17, ends with 17 and the sum of its digits is 17.

Result. 15317

Solution. We write the number in the form $100 \cdot a + 17$ for some $a \in \mathbb{N}$. Since 17 and 100 are coprime, we conclude that a is divisible by 17. Furthermore, a must have sum of digits 9 and therefore a is also a multiple of 9. The smallest a of this kind is $9 \cdot 17 = 153$, thus the solution is 15317.

Náboj 2014, problem 27

Problem. We call a prime p *strong* if one of the following conditions holds:

- p is a one-digit prime; or
- if we remove its first digit, we obtain another strong prime, and the same holds for the last digit.

For example, 37 is a strong prime, since by removing its first digit we get 7 and by removing its last digit we get 3 and both 3 and 7 are strong primes. Find all strong primes.

Result. 2, 3, 5, 7, 23, 37, 53, 73, 373

Solution. One-digit strong primes (1SP) are 2, 3, 5 and 7.

Two-digit strong primes (2SP) are prime numbers obtained by joining two 1SP together, namely 23, 37, 53, and 73.

Let us now find the three-digit strong primes (3SP): By removing the first digit we get a 2SP and similarly by removing the last digit. Therefore, we seek two 2SP such that the second digit of one is the first digit of the other. Out of the possible candidates 237, 537, 737 and 373, only 373 is a prime.

Finally, we focus on the four-digit strong primes (4SP). After removing both the first and the last digit we need to obtain 3SP, i.e. 373, which is clearly impossible. Hence 4SP do not exist and neither do strong primes with more than 4 digits. To sum up, the only strong primes are 2, 3, 5, 7, 23, 37, 53, 73, and 373.

Náboj 2018, problem 31

Problem. Written in the decimal system, the power 2^{29} is a nine-digit number whose digits are pairwise distinct. Which digit is missing?

Result. 4

Solution. On one hand, the power 2^{29} can be computed by hand with reasonable effort: For example, use $2^{10} = 1024$, compute 1024^2 and $1024^2 \cdot 1024$. Finally, divide the result by 2 to get $2^{29} = 536\,870\,912$.

On the other hand, you can use the fact that an integer and its digital sum have the same residue class modulo 9. Moreover, the residue class of 2^n modulo 9 is periodic with period of length 6. Since the sum of all digits is 45, we end up having

$$45 - x \equiv 2^{29} \equiv 2^5 \equiv 5 \pmod 9$$

where x denotes the missing digit in the decimal representation of 2^{29}. This leads to $x \equiv 4 \pmod 9$. Therefore the missing digit is 4.

Náboj 2017, problem 33

Problem. Write 333 as the sum of squares of (arbitrarily many) distinct positive odd integers.

Result. $3^2 + 5^2 + 7^2 + 9^2 + 13^2$

Solution. Since $17^2 = 289 < 333 < 361 = 19^2$, only the numbers $1^2, 3^2, \ldots, 17^2$ (altogether nine distinct numbers) may appear as summands. Further, whenever an odd integer is squared, the result gives remainder 1 when divided by 8—since 333 has remainder 5 after dividing by 8, the number of summands in the desired sum has to be five.

Since $1^2 + 3^2 + 5^2 + 7^2 + 17^2 > 333$, the square 17^2 cannot appear in the sum. Let us take a look at the leftover summands modulo 5: Two of them are divisible by 5 (5^2, 15^2), three of them give remainder 1 (1^2, 9^2, 11^2) and three give remainder -1 (3^2, 7^2, 13^2). Since 333 gives remainder 3 (or -2), there are two cases to consider. Firstly, we may sum up all the numbers with remainder 0 or 1; however, this turns out to exceed 333. Secondly, to achieve -2, we have to sum all the numbers with remainder -1, one with remainder 0 and one with 1. It is easy to see that the results containing 11^2 or 15^2 are too large, and out of the two remaining possibilities, only $3^2 + 5^2 + 7^2 + 9^2 + 13^2$ is equal to 333.

Náboj 2015, problem 34

Find all primes p such that $p + 11$ divides $p(p + 1)(p + 2)$.

Result. 7, 11, 19, 79

Solution. As p is a prime, it is either equal to 11 (which clearly satisfies the given condition) or coprime to $p + 11$. In the latter case, the product in question is divisible by $p + 11$ if and only if $(p + 1)(p + 2)$ is. This reduced product modulo $p + 11$ equals $(-10) \cdot (-9)$, thus $p + 11 \mid 90$. This is satisfied for $p \in \{7, 19, 79\}$.

Náboj 2019, problem 41

Problem. On a bicycle tour of length 110 km from Passau to Linz, Heiko and Eva have to overcome three climbs. During their first stop, Heiko, who is excellent at mental arithmetic, says: "If you multiply the three distances from Passau to the peak of each climb, you get a multiple of 2261." After thinking about that for a while, Eva responds: "You also get a multiple of 2261 if you multiply the distances measured from Linz instead of Passau." After riding 80 km from the start they make a second stop and Heiko remarks: "Now there is only one climb ahead before we arrive in Linz." Assuming that all the distances are integers and in kilometres, find the distances from Passau to the peak of each climb in kilometres.

Result. 68, 76, 91

Solution. Let A, B, and C be the three distances of the peaks from Passau measured in km. These distances have to satisfy $2261 \mid ABC$ and $2261 \mid (110 - A)(110 - B)(110 - C)$. Due to $2261 = 7 \cdot 323 = 7 \cdot 17 \cdot 19$ and $7 \cdot 17 = 119 > 110$ none of the distances can obtain more than one prime factor of 2261.

W.l.o.g. $7 \mid A$, $17 \mid B$, and $19 \mid C$. Since the same considerations apply to the distances $110 - A$, $110 - B$ and $110 - C$, we get the two possibilities $7 \mid (110 - B)$ and $7 \mid (110 - C)$ due to $7 \nmid (110 - A)$. In the case $7 \mid (110 - B)$ we get $19 \mid (110 - A)$ from $19 \nmid (110 - C)$, and finally $17 \mid (110 - C)$. In the case $7 \mid (110 - C)$ we get $17 \mid (110 - A)$ and $19 \mid (110 - B)$ in a similar way.

Since $\gcd(7, 19) = 1$, the only way to decompose 110 as $a \cdot 7 + b \cdot 19$ with a, b non-negative integers is $110 = 13 \cdot 7 + 1 \cdot 19$ (all the decompositions are of the form $110 = (13 + 19k) \cdot 7 + (1 - 7k) \cdot 19$ for $k \in \mathbb{Z}$ and the coefficients are non-negative only for $k = 0$). Similarly, we get the decompositions $110 = 4 \cdot 17 + 6 \cdot 7$ and $110 = 4 \cdot 19 + 2 \cdot 17$. This leads to the two solutions

$$A = 13 \cdot 7 = 91, \quad B = 4 \cdot 17 = 68, \quad C = 4 \cdot 19 = 76$$

and

$$A = 6 \cdot 7 = 42, \quad B = 2 \cdot 17 = 34, \quad C = 19.$$

Heiko's remark during the second stop indicates that the third peak is at least 80 km away from Passau. As a consequence, the sought distances are 68, 76, and 91.

Náboj 2019, problem 49

Problem. Find all integers $n \geq 3$ for which

$$\frac{(n-1)^{n-1} - n^2 + 2019 \cdot (n-1)}{(n-2)^2}$$

is an integer.

Result. 3, 4, 5, 6, 8, 14

Solution. We would like n to satisfy $(n-2)^2 \mid (n-1)^{n-1} - n^2 + 2019 \cdot (n-1)$. This is not influenced by adding $(n-2)^2$ to the right-hand side and it helps us to get rid of the term n^2, since $n^2 - 4n + 4 - n^2 = -4(n-1)$. We obtain an equivalent condition

$$(n-2)^2 \mid (n-1)^{n-1} + 2015 \cdot (n-1).$$

Since $n-1$ and $n-2$ are coprime, we can divide the right-hand side by $n-1$. Substitute $t = n-2$. Then $t^2 \mid (t+1)^t + 2015$. Finally, we use the binomial expansion to get

$$t^2 \mid t^t + \binom{t}{t-1} t^{t-1} + \cdots + \binom{t}{2} t^2 + \binom{t}{1} t + 1 + 2015,$$

so all what is needed is $t^2 \mid 2016$. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$, so the values 1, 2, 3, 4, 6, 12 are the only possible (and satisfying) for t. Substituting back into $n = t + 2$ leads to the result 3, 4, 5, 6, 8, 14.

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Changes in a society over twenty years are reflected in some mathematics challenge problems

Peter Bailey

Peter Bailey and others started the Primary Mathematics Challenge (UK) in 1999. He is a member of the Management and Problems Teams for this challenge. He worked as a teacher of mathematics in secondary schools in England and Africa. For a few years he was a member of the council for The Mathematical Association (UK). With Sharan-Jeet Shan in 1991, he wrote Multiple Factors: classroom mathematics for equality and justice. After leaving teaching, he was a bus driver and then a housing officer for asylum seekers.

Abstract
The Primary Mathematics Challenge (PMC) is run by The Mathematical Association (MA) in the UK. It is aimed at school children aged eleven and under. The national challenge aims to interest and motivate young people into mathematics by using a mixture of pure and real-life problems, with some humour and consideration of simple ethical issues. In 2005 there was an article about the PMC in Mathematics Competitions. Some of the changes in culture that have taken place since the challenge started are shown in this paper.

Introduction

The PMC is set in November each year, with answers and notes provided for the teachers. It is a 25-question multiple choice challenge which is not tied to the UK National Curriculum. Approximately 1500 schools buy packs of 10 PMC papers. The papers are marked in school teachers. Certificates are awarded by the teachers for presentation in school assemblies. Pupils who achieve top scores in November take a harder Bonus Round challenge in February. Their answer sheets are marked by the MA and medals awarded to those scoring high marks. Sample papers are available at www.primarymathschallenge.org.uk. The paper is structured as follows. After a short summary of initial and actual research on test-wiseness, the taxonomy by Millman et al. (1965) will be connected to the format of the Mathematical Kangaroo. The execution of the KATS-questionnaire and the following evaluation serve as a foundation for the final discussion of the strategies concerning the principles of test-wiseness at the Mathematical Kangaroo, and an outlook on further possible research is given.

Real-life contexts in the PMC

Often primary mathematics problems are either abstract or about very simple real-life situations such as 'mummy goes shopping'. Pure maths is very important. But opportunities for more serious real-life issues are often avoided, and pupils do not have the opportunity to see mathematics being used in a wide range of human activity.

The PMC aims to motivate as many primary children as possible in mathematics. Some will turn out to be pure mathematicians, others will work in banks, businesses, IT, and will be designers, or running their homes etc. Many PMC problems are about abstract mathematics such as Question 1.

Question 1 a and b are two different whole numbers less than 10. If $a^3 = b^2$, what number could a be?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 8

But some problems in the PMC are set in real life situations. Most mathematics examinations and challenges avoid setting real-life problems in situations which allow moral issues to be discussed. It is felt that the mathematics classroom is not the place for that. However mathematics is fundamental to describing our world and in looking at the problems we face. To avoid using mathematics in these situations is to reduce pupils' perceptions of mathematics as an agent for change and for good (or bad). A guidance document for members of the PMC Problems Team says that a good PMC paper will have some questions based on pure mathematics and others on many varied practical real-life contexts. These will show how mathematics plays an important role in life and may also have an ethical / moral aspect.

Some changes in the UK culture

In the twenty years of the PMC, culture has been changing. PMC problems have also changed and some of the major developments in our society can be seen in the challenge papers. Here are a few examples.

The National Lottery was going well in 2002, and at that time primary children in the UK were being taught probability. Question 2 uses simple words to describe the chances of winning the lottery.

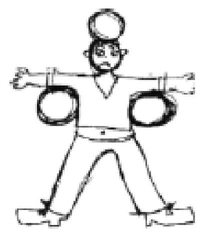
Question 2 A shop sells National Lottery tickets. You see the sentence 'It could be you'. What is the chance that someone, with one ticket, will win a large prize with a National Lottery ticket?
(A) impossible (B) very unlikely (C) unlikely (D) evens (E) likely

Again in 2002 Mobile phones were everywhere and the pressure on parents to provide each child with one must have been huge. Question 3 is about a mobile phone number.

Question 3 Maisie was telling me her mobile phone number but she sneezed in the middle. She said it was divisible by 9. I got her number as 07922?03441. Which of these numbers could replace the unknown number?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

There was also another problem on probability in which pupils could list all the equal possibilities to get the answer. Question 4 could lead to a classroom discussion on calculating the probability.

Question 4 Here is terrified Terry. His girlfriend Miss D Meanor says she can burst all 3 balloons with 3 arrows. The chance of each arrow bursting a balloon is $\frac{1}{2}$. What is the probability that all 3 balloons are burst with 3 arrows?
(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) $1\frac{1}{2}$



The first problem concerned with the environment was Question 5 and was set in 2005.

Question 5 Two cars make the same journey of 240 miles. One car does 40 miles per gallon (mpg) and the other 30 mpg. How many gallons of fuel are saved by using the first car?
(A) 2 (B) 6 (C) 8 (D) 10 (E) 14

By 2006, pound shops had opened in many towns. In my suburb of Birmingham, Poundland opened, followed by The 99p Shop and then by 98p - why pay more? This shop only lasted a few weeks. Question 6 is about cut-price shopping

Question 6 99 Pence Shop Company has all goods priced at 99p. How much do 7 cost?
(A) £0.93 (B) £0.99 (C) £6.93 (D) £7.93 (E) £7.99

By 2007 school league tables, publishing the examination results for all schools in the UK, were making some teachers feel that they must concentrate on teaching to the exams and 'stretch' their pupils more and more.

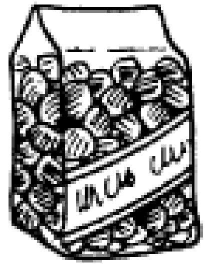
Question 7 Teachers like 'stretching' pupil by giving them hard maths problems. Mona Lotte is stretched from 160 cm to 176 cm. What percentage stretching is this?
(A) 6% (B) 10% (C) 16% (D) 60% (E) 76%

In 2008 people were buying low energy light bulbs to reduce use of electricity - see Question 8.

Question 8 In my room there was one 100 watt light bulb which was on for 5 hours a day. I changed it for a 20 watt low-energy bulb and use it for only 4 hours a day. What percentage of watts is saved each day?
(A) 20 (B) 75 (C) 80 (D) 84 (E) 86

2009 saw healthy eating becoming important for parents and for school meals Question 9 asks pupils to think about how much fat is in a snack.

Question 9 You have a 350g pack of dried fruit snack. The guideline daily amount (GDA) of fat is 70g. The fruit snack contains 2g fat in 100g. What percentage of the GDA does the pack contain?
(A) 2% (B) 7% (C) 10% (D) 20% (E) 70%



Perhaps we need to be sensitive about some healthy eating questions. In 2019 a few pupils protested about a mathematics exam question. It asked how many calories a woman had consumed for breakfast. This was found to have been distressing for a few candidates with an eating disorder. One girl had to leave the exam hall in a panic (Daily Telegraph 11 June 2019).

Saving water was highlighted in Question 10a.

Question 10a On average a person visits the toilet 2500 times a year. Older toilets use 9 litres of water per flush but more modern ones use only 6 litres per flush. How many litres of water would a family of four save in a year by using a new toilet instead of an old one?
(A) 3 (B) 3 000 (C) 7 500 (D) 30 000 (E) 3 million

By 2010 coffee culture was spreading with cafés opening up in every high street. Question 10 prompted a feedback comment from one pupil: 'My Mom drinks coffee in Costa Packet'.

Question 10b Costa Packet pays 90p for a coffee in her local coffee bar. If she buys five cups of coffee she gets one extra cup free, so how much does she effectively pay for each cup of coffee?
(A) 15p (B) 75p (C) 90p (D) £4.50 (E) £5.40

In 2011, Bonuses of bankers and others were in the news with a few bosses in the UK earning ridiculous sums of money. Question 11 implies that they were just plain greedy.

Question 11 A city banker gets a bonus of £2 million. He says, 'I will give my five children £1 000 each, and my wife £5 000'. How much does he keep for himself?
(A) £199 (B) £1990 (C) £9900 (D) £199000
(E) £1990000

Also in 2011 many organisations were aiming to reduce energy consumption. Question 12 concerns a school planning to reduce its energy use.

Question 12 Our school has agreed to reduce its energy use over the next three years. Each year, the amount of energy used will be 10What will be the total percentage reduction in energy use over the three years?
(A) 27.1 (B) 30 (C) 70 (D) 72.1 (E) 100

2011 also saw stop and search by police as a problem for some sections of society and this is reflected in this Question 13.

Question 13 Yesterday the police asked five schoolgirls for their dates of birth. Which of them did not tell a lie? (This question was set in November 2011)
(A) Neoprene (17.13.2001) (B) Nylon (32.05.2000)
(C) Formica (14.07.2012) (D) Teflon (16.08.2001)
(E) Trogamid (30.02.2002)

The small number of people voting in elections was also causing concern with between 30 and 40% of the electorate taking part in some elections (Bogdanor 2011). Question 14 highlights the small number of citizens which can, in low voting turnout, win an election.

Question 14 In an election, only 50% of the citizens voted. Of these, 60% voted for the winning party. What percentage of the citizens voted for the winning party?
(A) 30% (B) 35% (C) 40% (D) 45% (E) 50%

Interestingly, in the UK 51.89% voted leave in the 2016 EU referendum on a 72.21% registered voter turnout, giving 37.47% of registered voters voting to leave the EU (Wikipedia). Many people did not register to vote. In all, 17 410 742 voted to leave, out of a population of 65.6 million (Office for National Statistics). This is mentioned because a short PMC question can raise important factors in the reality of our democracy.

Around this time the colour of the aisles with girls' toys in Woolworths was predominantly pink, whereas the boys' aisles were more macho and in blues and greys. Pinkstinks was a campaign founded in 2008 to raise awareness of what they claim is damage caused by gender stereotyping of children. The figures in Question 15 were taken from a survey at the time and provide an opportunity to think about this issue.

Question 15 The table shows the results of a survey in which 60 girls and 60 boys were asked what colour of dress girls should wear. What percentage of boys thought girls should wear pink?
(A) 5% (B) 10% (C) 20% (D) 40%
(E) 100%

	blue	pink	black
girls	12	6	42
boys	15	24	21

How much different workers should get paid is always an issue. A question on relative pay highlighted this. Question 16 was a difficult problem with ratios and was set in the Bonus

Round in 2013.

Question 16 A cleaner earns £16 000 a year, a teacher earns £28 000 a year and a banker earns £72 000. Suppose they all need £12 000 to pay the basic costs of living. If they all take away the basic costs of living from their salaries, which of the following ratios (cleaner : teacher : banker) best describes the money they are left with for other spending?
(A) 1:1:1 (B) 1:2:3 (C) 1:4:15 (D) 4:7:18 (E) 16:28:72

DVD players became available in 2000. In my time recorded music has been on vinyl, tapes, CDs, solid state and now can be down-loaded, an amazing series of developments. Question 17 mentions the availability of DVD films in 2013.

Question 17 In 2011, a quarter of a million DVDs of Doctor Why were sold. In 2012, the number of sales increased by 10%. How many people bought a Doctor Why DVD in 2012?
(A) 25 000 (B) 27 500 (C) 250 000 (D) 275 000 (E) 2 750 000

By 2013 pupils were being encouraged to walk or cycle to school for their own health and to save congestion on the roads around schools. Question 18 is about average speed.

Question 18 My name is Speedy. It takes me 3 minutes to cycle to school, which is half a mile away from my home. What is my average cycling speed?
(A) 4 mph (B) 6 mph (C) 8 mph (D) 10 mph (E) 12 mph

The Bangladesh government was the first to take action in reducing the use of lightweight plastic bags. By 2014 problems of single use plastic were being raised. England was the last country in the UK to adopt the 5 pence charge, with the levy taking effect on 5 October 2015. Question 19 shows how many plastic bags can be saved by even a small town,

Question 19 Polly Bagg lives in a small town of 500 people. Every day each person gets one polythene bag from the local shop. If all the people in the town decided to use their own bags instead, roughly how many bags would be saved in a year?
(A) 500 (B) 1800 (C) 18 000 (D) 180 000 (E) 1.8 million

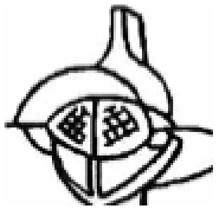
Climate Change was in the news in 2015 and the effects on butterfly life mentioned. Question 20 takes information about the comma butterfly which was reported in the

Intergovernmental Panel on Climate Change (2013) and which provides data on species migration.

Question 20 The places you might expect to find certain butterflies are moving 5 m northwards every day, perhaps because of climate change. Roughly how far northwards will such butterflies have moved in one year?
(A) 5 m (B) 60 m (C) 260 m (D) 365 m (E) 1.8 km

In 2013 UK Education Secretary Michael Gove removed the teaching of probability from the primary curriculum for England and replaced it with the teaching of Roman Numbers to 1000 (M) and recognition of years written in Roman numerals (Year 5 Programme of Study 2013). In 2015, Question 21 used Roman numbers.

Question 21 In Roman numerals, 2015 is written using 4 letters: MMXV. In how many years' time will the year next be written with only 4 letters ?
(A) 4 (B) 5 (C) 6 (D) 9 (E) 10



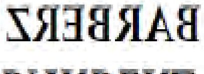
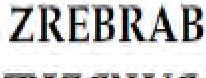
The disappearance and reduction of wild life species were again in the news. In 2016 Question 22 used figures for the fall in the number of African lions, decreasing by 90% between 1975 and 2015.

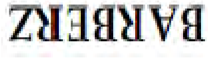
Question 22 In 1975 it was estimated that there were 250 000 lions in Africa. Over 40 years this figure has decreased by 90%. What is the current estimate for the number of lions in Africa?
(A) 25 000 (B) 100 000 (C) 160 000 (D) 225 000 (E) 275 000

By 2017 pupils and the rest of us were getting familiar with download speeds for computers and phones - see Question 23.

Question 23 Which of the following progress bars indicates 23.4 MB of 37 MB downloaded?
(A)  (B)  (C) 
(D)  (E) 

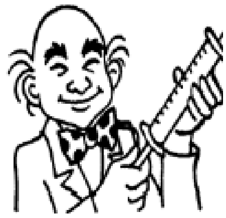
Also in 2017, Question 24 reflects the huge rise in the number of barbers on the high streets.

Question 24 Jess could see that someone had printed the words BARBERZ on the window. How did it look from the other side of the window?
(A)  (B) 

(C)  (D) 
(E) 

In 2018 doctors were being expected to work harder and **National Health Service targets** were in the news. Question 25 highlights flu jabs in the autumn for a large section of the population - this is now the norm.

Question 25 Dr Whynot Jabbemall has to jab 300 patients in 30 minutes. In the first ten minutes he jabs at a rate of 6 per minute. How many jabs per minute must he make in the next 20 minutes to compete his task?
(A) 6 (B) 12 (C) 15 (D) 20 (E) 30



In 2019 Question 26 was about app. By then it seemed reasonable to assume that primary children know what an app is.

Question 26 Agnijo has half as many apps as Sam who has a third as many apps as Naomi. Altogether, they have 180 apps. How many apps does Sam have?
(A) 20 (B) 30 (C) 40 (D) 60 (E) 90

And with the rise of Extinction Rebellion in 2019/20, there may be another question on the problems facing our environment in next year's PMC papers.

However not all 'real-life' problems in the PMC are practical. We can easily do noughts and crosses in three dimensions and can use imagination to consider problems like Question 27.

Question 27 Imagine a world which can have a negative number of people. A bus in this world had 5 passengers. Then nine got off. Then six more got on. How many passengers are on the bus now?
(A) -20 (B) -2 (C) 0 (D) 2 (E) 20

Finally, with all these real-life questions, our intention is not to worry children unduly. But by the age of ten and eleven years, a good education should provide some food for thought. By including examples of how mathematics can explain and be used to improve our world, children will see the fullness of mathematics, both pure and applied.

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Costa Rican Mathematics Olympiad for Elementary Education - OLCOMEP

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As a professor of didactics of mathematics, she prepares future elementary teachers at the University of Costa Rica (UCR) and future mathematics' teachers at Universidad Estatal a Distancia de Costa Rica(UNED). She received the Master degree in Didactics of Mathematics by the University of Granada, Spain. Currently, she is doing her doctoral thesis about mathematical talent in the University of Valencia. Teacher and researcher. Developing research projects in the Institute of Psychological Investigations(IIP) and in the Center of Investigations on Mathematics and Metamathematics(CIMM) both investigations centers at the UCR. Become member of the central commission of OLCOMEP since 2018. She is the founding president of the association for the promotion of mathematical talent in Costa Rica (APOTEMA).

Abstract
This essay aims to describe the national mathematical competition carried out in elementary education in Costa Rica. A brief review of its origin, the way in which its general logistics is organized, the impact it has had on the student population, as well as some more specific aspects of the test structure and the characteristics of the items are described. Finally, some examples of exercises from the different stages in previous years are presented.

Origin and Organization

The National Olympiad of Mathematics for Elementary Education OLCOMEP[22], began in 2015 as a national initiative of the National Mathematics Advisory for Elementary Education. Its origin is really located in the Puriscal regional advisory, under the responsibility of Mr. Javier Barquero Rodríguez, who organized regional olympics in 2003 in response to the students of the region who wished to participate in mathematics olympics, in such a way that they lived a regional process that collaborated in their mathematical formation and enjoyment for mathematics. On that Puriscal regional olympiad, there was an approximate participation of 180 students from the region, 30 for each school level (from first to sixth grade). After the success of this regional competition, the idea was

[21]IIP for its name in Spanish: Instituto de Investigaciones Psicológicas
[21]CIMM for its name in Spanish: Centro de Investigación en Matemática y Meta Matemática
[21]APOTEMA for its name in Spanish: Asociación para la promoción del talento matemático
[22]For its name in Spanish: Olimpiada costarricense de matemática en educación primaria

replicated at national scale since 2015. Then, OLCOMEP becomes stronger, having from 2018 the advice of specialists from the Universidad de Costa Rica(UCR).

Being an activity organized by the Ministerio de Educación Pública (MEP), it is free of charge and its structure is similar to MEPs organization, that is why the olympiad has 4 phases of competition: the first phase is at school level, the second at the departmental level, third at the regional level and final stage at national level. All the phases are scheduled in the school calendar. Participants are grouped by their school grade, so olympiad consists of six simultaneous competitions, one for each grade of elementary education. The purpose of this competition is to stimulate and develop abilities to solve mathematical problems in children, through healthy competition between students from different educational regions. Students from first to sixth year of elementary education from public and private schools in the country are invited on equal terms to participate in competition. They have to solve mathematics problems that are not conventional, these are not like the problems that students make in their math classes. Because these are designed with the specific goal of identifying the mathematical talent so required a group of skills associated to that.

Participation

Participation in the OLCOMEP olympiad is massive, and has been increasing every year. Table 1 shows the number of participants from 2015 to the present. As shown, the growing interest of the educational community to participate in this activity to promote mathematical talent is eminent. The phases are organized like the MEP administrative structure, that is why we work with de core that are schools, a group of schools make a department or circuit. Then a group of departments make a region. Finally, the entire country is divided in 27 regions (in educational terms by de MEP).

Table 1. Students in the OLCOMEP period 2015-2019

Year	**Participants**	**Students by phases**		
		Departamental	**Regional**	**National**
2015	5	1880	220	90
2016	11	5800	425	132
2017	20	10200	3600	225
2018	25	11450	*	*
2019	25	11800	3650	285

* The competition could not be finished by a national strike.

The olympiad has significant implications in the country and participation is expected to increase even more, because there are 27 educational regions in the country, so there are still some that do not have representation in the olympiad. Every year it motivates more students and we spread the information all over the country to have a huge coverage. OLCOMEP has the backing of the MEP and the UCR, it is coordinated by a central commission, conformed by two representatives of the MEP and two of the UCR, who are responsible for the logistics, academic quality, development and application of the olympiad, and it also has the support of regional advisors and teachers involved in it.

The results of this process start to be evident when some elementary school students who participate in the OLCOMEP olympiad continue their participation in the national high school olympiad (OLCOMA), reaching medals and honorable mentions in them. Furthermore, some of them have obtained medals and honorable mentions in international Olympics such as the olimpiada de mayo, at the IMO, and have also participated in the EGMO 2018 (European Girls' Mathematical Olympiad). In 2016, a Costa Rican student obtained a gold medal at the IMC (International Mathematics Competition for University Students), which is the first gold medal obtained by Costa Rica. This student attended public schools, both at elementary and secondary education, and participated at the first regional mathematics olympiad in 2003.

Test structure

The tests are aligned with the national mathematics curriculum (MEP, 2012), promoting the development of skills in participating students and to generate positive attitudes and beliefs towards the teaching and learning of mathematics, which aims to contribute to the improvement of the educational quality of the country. That is why the results obtained in the applied tests are analyzed and each region is fed back with these data. Trainings are given to teachers involved throughout the competition year, so it intends to direct these trainings to the areas whose results have been low in previous editions. Therefore, it provides opportunities for those students who have talent in the area of mathematics and needs to continue developing these skills.

Currently the competition is focused on identifying mathematically talented students, that is, those having some specific skills including flexibility, originality, ability to generalize, visualize, or transfer. It is intended that students demonstrate in the tests through the resolution of mathematical problems and to allow us to identify the most talented students. That is why the tests do not have routine exercises like those usually solved in classrooms or textbooks; rather they have exercises that respond to mathematical talent, which allow the student to show some of those components of mathematical talent. Thus, the problems must satisfy certain criteria in the construction process and also must be submitted to a judgment process by the other members of the organizing commission to decide if it fits to the desirable characteristics (construct, skills, content, complexity, format, representations, context, resolution time), that is why the construction of items follows a somewhat rigorous process, to ensure that they correspond to the test construct. It is important to quote that the exercises must be solvable by the students with the mathematical contents that, according to the study program, the student manages until each phase, which includes contents that has been studied in that course or previously.

The tests are composed by 10 to 12 short response exercises and 3 to 4 open-ended problems (depending of the grade). They are distributed into three levels of complexity (reproduction, connection and reflection)[23] and they are also distributed into the four areas in the mathematics curriculum, geometry, algebra and relations, measures, and statistics and probability, as well as distributed in the different contexts in which the problems posed may be located (real, fictitious, mathematical) as presented in the national curriculum.

Each year, practice booklets are prepared with the objective that students can train for the competition. There is a booklet for students and one for teachers of each educational level (bringing the solved exercises, with various solution strategies, step by step solutions, and explanations), all this with the objective that students and teachers have material to be

[23](MEP, 2012, p.33-34)

prepared to this competition, since an objective is that students have chance to learn and develop their capabilities. For the assembly of the tests, a table is made to ensure that the problems have the previously established distribution of levels of difficulty, content areas, and contexts. Those percentages (table 2) have been defined by the central commission of OLCOMEP, and which respond to the distribution proposed by the MEP in terms of contexts and content areas. The level of difficulty of the problems has a progressive increase from one phase to the next, reaching a minimum level of reproduction problems and a higher level of reflection problems in the final phase, with the purpose that the students can participate and initially go on, but as the competition goes further the students can be selected according to their mathematical talent.

Table 2. Percentages of content areas, complexity and context by phases.

	Grade:	**1st**	**2nd**	**3th**	**4th**	**5th**	**6th**
Content areas	*measures*	10%	10%	10%	10%	5%	5%
	numbers	45%	45%	40%	35%	30%	25%
	algebra and relations	15%	15%	15%	15%	25%	30%
	geometry	20%	20%	25%	25%	25%	25%
	statistics and probability	10%	10%	10%	15%	15%	15%
					phases		
			II		**III**		**IV**
Complexity	*Reproduction*		50%		40%		20%
	connection		40%		40%		50%
	reflection		10%		20%		30%
Context	*real*				20%		
	fictitious				30%		
	mathematical				50%		

Each year, six tests are built for each phase (one per school grade), which makes a total of 24 tests per year. Problems of phases I, II and III are released every year after the results of each phase are communicated, but the problems of the final phase are maintained confidentially and take part of a bank of items.

Examples of problems

Below I present some examples of development exercises that were part of the 2019 competition tests, as well as some possible solution strategies. These strategies were considered the appropriated for the capabilities and knowledge of the students in the correspondent school grade.

Third year. Phase II. 2019. Exercise 3.
A box contains a quantity of chocolate bars satisfying the following conditions:

- If they are divided into groups of three, there are two bars left over.

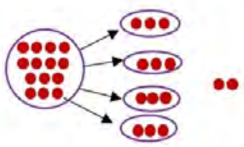
- They can be divided into two equal groups and there are none left over.

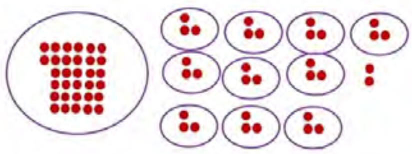
- The sum of the two digits of the number of chocolate bars is 5.

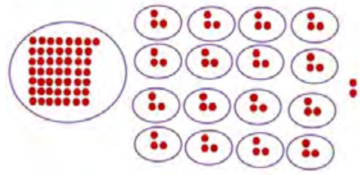
How many chocolate bars are there in the box? Justify your answer.

Possible solution, strategy 1
The student can start looking for the two-digit numbers whose digits add up to 5, to narrow the search. The numbers would be 14, 23, 32, 41 and 50. Numbers 23 and 41 are discarded because they do not match the second condition. So only numbers 14, 32 and 50 should be checked against condition 1.

*Test with number 14: *Test with number 32 *Test with number 50:







All the three numbers satisfy condition 1, because there are two chocolates left over. So, we can have 50, 32 or 14 chocolate bars in the box.

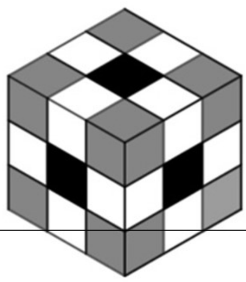
Possible solution, strategy 2
Like in strategy 1, students can start identifying the numbers 23, 32, 41, 14, 50. Now, after checking the first condition, none is eliminated, because all numbers satisfy that (from the previous list):

$$\begin{aligned} 23-2=21 \text{ and } &\quad 21=3\times 7\\ 32-2=30 \text{ and } &\quad 30=33\times 10\\ 41-2=39 \text{ and } &\quad 39=33\times 13\\ 14-2=12 \text{ and } &\quad 12=33\times 4\\ 50-2=48 \text{ and } &\quad 48=33\times 16 \end{aligned} \tag{4}$$

$$\tag{5}$$

Finally, when checking the second condition, 23 and 41 are eliminated because they cannot be divided into two equal groups. So, there may be 32, 14 or 50 chocolate bars in the box.

Fourth year. Phase II 2019. Exercise 3.
Sofia is playing with small cubes and built a big cube. The big cube has each face made with cubes of three different colors, as shown in the figure.



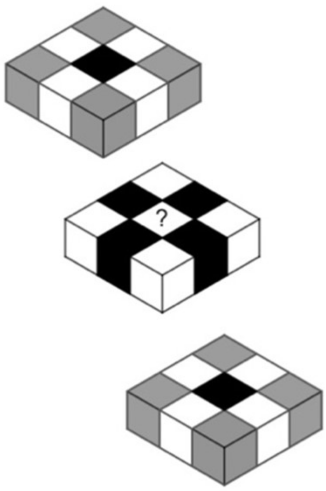
If Sofia tries to build the cube so that the upper face only has cubes of the same color, could she do it so that the upper face is completely black? And gray? Justify your answers.

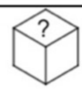
Possible solution strategy 1
Calculate the number of small cubes of each color, noting that being the six faces of the cube identical then it has:

6 black cubes	one in each face
8 gray cubes	one in each corner
12 white cubes	four in each plane of the cube

As 9 cubes are needed to make the upper face, it cannot be completely black. However, considering that there is a small cube in the center of the big cube whose color is unknown, if that cube was gray, the big cube could be built with the upper face completely gray.

Possible solution strategy 2
The cube is disassembled by planes and the number of little cubes of each color is calculated:



	6
	8
	12
	1

There is a little cube whose color is unknown. To complete the top face, 9 cubes are needed. Then, the face cannot be completely black. If the cube in the center, of which we do not know the color, was gray, the upper face could be completely gray.

Fifth year. Phase II 2019. Exercise 1.

Roberto has several square pieces of wood. He assembles the pieces to make figures shown, by following a rule invented by him.

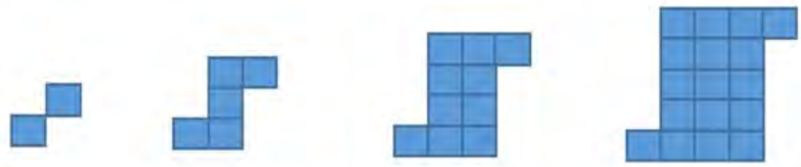


Figure 1 Figure 2 Figure 3 Figure 4

a. How many square pieces does Roberto need for Figure 9?

b. How can you explain the rule used to calculate the number of pieces of any figure?

c. Which figure can you built with 442 pieces, using all the pieces?

Possible solution strategy a.1
The student continues drawing the figures

Figure 5 Figure 6 Figure 7 Figure 8 Figure 9

And then he counts the pieces of figure 9, 82 pieces.

Possible solution strategy a.2
The student deduces that each figure is formed by adding to the previous one a prime number:

Number of figure	Process	Amount of pieces
1		2
2	2+3	5
3	5+5	10
4	10+7	17
5	17+9	26
6	26+11	37
7	37+13	50
8	50+15	65
9	65+17	82

Possible solution strategy a.3
The student visualizes the cube, noticing that there is a rectangle whose base is a unit smaller than its figure number and whose height is a unit larger than the figure number, and two squares are added on the sides. So, figure 9 would have a rectangle with base 8

and height 10, that is 80 pieces, and the two other pieces, so figure 9 has 82 pieces.

Possible solution strategy a.4
The student observes the following relationship between the number of pieces and the figure number:

Number of figure	Process	Amount of pieces
1	$1 \times 1 + 1$	2
2	$2 \times 2 + 1$	5
3	$3 \times 3 + 1$	10
4	$4 \times 4 + 1$	17
5	$5 \times 5 + 1$	26
6	$6 \times 6 + 1$	37
7	$7 \times 7 + 1$	50
8	$8 \times 8 + 1$	65
9	$9 \times 9 + 1$	82

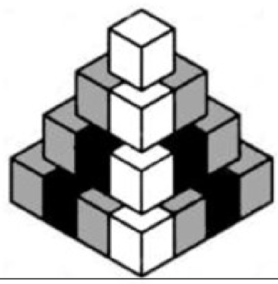
Possible solution strategy b.2
"The figure number is multiplied by it and one is added"

Possible solution strategy b.3
"The number of the figure is multiplied by two and then rest one, obtaining a prime number, that prime number is added to the number of pieces in the previous figure".

Possible solution strategy c.1
Relying from option b.1 of the previous question: 442-2 = 440 And then to think two numbers that multiply make 440 and be the before and after of a certain number (that number would be the figure number) 20 x 22 = 440. Then it is figure 21.

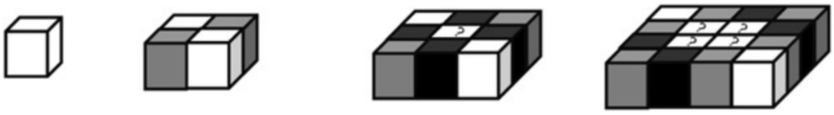
Possible solution strategy c.2
Relying from option b.2 of the previous question: 442-1 = 441 And then find a number (figure number) that multiplied by it make 441; (21x21 = 441) so that figure 21 can be formed.

Fifth year. Phase III 2019. Exercise 3.
Fabio's father built a structure with cubes to decorate the living room. Fabio notes that looking at the structure through any of the corners looks identical. After his father paints the visible cubes with three different colors, Fabio notes that identical views are now achieved by observing the structure from any corner and its opposite, but the views of the two pairs of corners are different. Look at the painted structure:



a. How many of the visible cubes did he paint black? Justify your answer. b. How many of the visible cubes did he paint white? Justify your answer. c. How many of the cubes that make up the structure were not painted? Justify your answer.

Possible solution strategy 1
You can disassemble the figure by floors.



Noting that there are:

- Seven white cubes.
- Ten gray cubes.
- Eight black cubes.
- Five unpainted cubes.

Fourth year. Phase III 2019. Exercise 1.
I am a number containing 4 different digits. I satisfy the following conditions:

- The digit of the units is three times the digit of the thousands.
- The digit of the tens is the product of the digit of the thousand and the digit of the hundreds.

Which are the numbers satisfying those conditions?

Possible solution strategy 1
Analyzes the possible answers by reading the conditions in the given order:

- The digit of the units is three times the digit of the thousands.

Th	**H**	**T**	**O**
1			3
2			6
3			9

Then, check which of these combinations satisfy the second condition:

- The digit of the tens is the product of the digit of the thousand and the digit of the hundreds.

	Case 1				**Case 2**				**Case 3**			
	Th	**H**	**T**	**O**	**Th**	**H**	**T**	**O**	**Th**	**H**	**T**	**O**
	1			3	2			6	3			9
Posible	~~1223~~				~~2126~~				~~3139~~			
answers	~~1333~~				~~2246~~				3269			
	~~1443~~				~~2366~~				~~3399~~			
	~~1553~~				2486							
	~~1663~~											
	~~1773~~											
	~~1883~~											
	~~1993~~											

As the four digits must be different, only 2486 or 3269 fit all the conditions.

Possible solution strategy 2
Start working with the second condition, but considering at the same time that the digits must be different:

- The tens digit is the product of the thousand units' digit and the hundreds digit.

Th	**H**	**T**	**O**
2	3	6	
2	4	8	
3	2	6	
4	2	8	

Then check which of these combinations can satisfy the first condition:

- The digit of the units is triple the digit of the thousand units.

Th	**H**	**T**	**O**
2	3	6	6
2	4	8	6
3	2	6	9
4	2	8	12

As the four digits must be different, 2486 and 3269 are the solutions.

Reference

Ministerio de Educación Pública (MEP), *Reforma curricular en ética, estética y ciudadanía. Programas de estudio de matemáticas. I y II Ciclo de la Educación Primaria, III Ciclo de educación general básica y educación diversificada* (2012)

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Which test-wiseness based strategies are used by Austrian winners of the Mathematical Kangaroo?

Lukas Donner, Jakob Kelz, Elisabeth Stipsits, and David Stuhlpfarrer

Lukas Donner (née Andritsch) is a former participant of national and international mathematical competitions and has now been working in the field of Mathematical Olympiads for several years by giving regular classes for interested students and by being part of committees of mathematical competitions. He finished his PhD in pure maths in February 2019 and after postdoctoral positions at both the University of Graz and the University of Vienna he is now working at the University of Duisburg-Essen. Beside combinatorial algebra, his main research interest lies in mathematical education, especially in problem solving and mathematical competitions.

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Abstract
Test-wiseness describes the usage of strategies, which support successful responses on multiple-choice tests, independent of the knowledge of the underlying topic. Due to the construction of the Mathematical Kangaroo, it is suitable for applying test-wiseness strategies. The strategies were formulated based on the test-wiseness guiding strategies and merged into a specially developed KATS (KAngaroo-Test-wiseness-Strategies) questionnaire. This questionnaire was presented to the Austrian winners of the Mathematical Kangaroo 2018, grades 3 to 13. The findings from this study provide on the one hand information on preferred strategies (top-ranked strategies), and on the other hand how this particular group prepares for the Mathematical Kangaroo.
Keywords: Test-wiseness, Multiple-Choice, Mathematical competition, Mathematical Kangaroo

Introduction

Test-wiseness describes the capability to use specific strategies which are independent of thematic knowledge, in order to have advantages at multiple-choice tests. Its application can have a significant effect on success in multiple-choice formats. This has been investigated by many authors (e.g. Thoma und Köller 2018, Chang 2007 Doly & Williams 1986). Due to its test-construction, the Mathematical Kangaroo can be connected perfectly with the theoretical concept of test-wiseness by Millman et al. (1965). Based on the principles of test-wiseness the authors created the KATS (KAngaroo-Test-wiseness-Strategies)-questionnaire. It contains the relevant parts of the guiding strategies of the principles of test-wiseness for the Mathematical Kangaroo. The items within the questionnaire are formulated in a way that is comprehensible for students. The strategies within the KATS-questionnaire, formulated as items, were investigated by evaluating the replies of the Austrian winners of the Mathematical Kangaroo of grades 3 to 13. Within each test-wiseness principle, those strategies with great importance for the pupils have been identified. In the course of the investigation, the top-ranked strategies of that group have been compared to the less important strategies. Furthermore, the composition of the group of the Austrian winners has been investigated with respect to their preparation for the Mathematical Kangaroo. A further discussion had its focus on the relevance of the different principles of test-wiseness from the perspective of the Austrian winners.

The paper is structured as follows. After a short summary of initial and actual research on test-wiseness, the taxonomy by Millman et al. (1965) will be connected to the format of the Mathematical Kangaroo. The execution of the KATS-questionnaire and the following evaluation serve as a foundation for the final discussion of the strategies concerning the principles of test-wiseness at the Mathematical Kangaroo, and an outlook on further possible research is given.

Test-wiseness

Test-wiseness and its principles

Test-wiseness was developed by Millman et al. (1965). It is defined as the ability to advantageously use different strategies to correctly solve as many multiple-choice tasks as possible, regardless of the thematic knowledge. In connection with test-wiseness, the use of various strategies to answer multiple-choice tasks is important (Thoma & Köller 2018; Tomkowicz & Rogers 2005; Smith 1982; Millman et al. 1965; Gibb 1964). Millman et al. (1965) present a taxonomy of test-wiseness principles for capturing test-wiseness in two main categories as follows (see Table 1): Principles that are independent of the test constructor or test purpose and principles that are dependent upon the test constructor or purpose. The time-using and error-avoidance strategies are assigned to the principles of the first category. Time-using strategies only apply to test situations with time constraints; however, error-avoidance strategies are used in all test situations. Both strategies enable the tested participants to demonstrate their knowledge of specific topic content, whereby loss of points due to knowledge deficits can be avoided. In contrast, guessing strategies and deductive reasoning strategies in this category allow the test participants to gain points with their knowledge that go beyond their thematic knowledge.

The second category includes strategies that are dependent on the test purpose and the test developer. Similar to the first two subdivisions in the previous category, the intent consideration strategies make it possible to avoid deduction points due to a lack of knowledge. The cue-using strategies are available to the participants if a specific answer is not known or only partial knowledge is available. However, the successful use of the information usage strategies depends on the extent to which there is a thematic connection between the information and the correct answer (Millman et al. 1965). The following table shows the test-wiseness principles according to Millman et al. (1965), with the guiding strategies in italics to which the KATS questionnaire does not refer, since these strategies are not compatible with the format of the Mathematical Kangaroo (see below for details).

Table 1: Taxonomy of test-wiseness nach Millman et al. (1965)

I. Elements independent of test constructor or purpose

A: Time-using strategy

1. Begin to work as rapidly as possible with reasonable assurance of accuracy.
2. Set up a schedule for progress through the test.
3. Omit or guess at items (see I.C. and II.B) which resist a quick response.
4. Mark omitted items, or items which could use further consideration, to assure easy relocation.
5. Use time remaining after completion of the test to reconsider answers.

B: Error-avoidance strategy

1. Pay careful attention to directions, determining clearly the nature of the task and the intended basis for response.
2. Pay careful attention to the items, determining clearly the nature of the question.
3. Ask examiner for clarification when necessary, if it is permitted.
4. Check all answers.

C: Guessing strategy

1. Always guess if right answers only are scored.
2. Always guess if the correction for guessing is less severe than a ”correction of guessing” formula that gives an expected score of zero for random responding.
3. Always guess even if the usual correction or a more severe penalty for guessing is employed, whenever elimination of options provide sufficient chance of profiting.

D: Deductive reasoning strategy

1. Eliminate options which are known to be incorrect choose from among the remaining options.
2. Choose neither or both of two statements which imply the correctness of each other.
3. Choose neither or one (but not both) of two statements, one of which, if correct, would imply the incorrectness of each other
4. Restrict choice to those options which encompass all of two or more given statements known to be correct.
5. Utilize relevant content information in other test items and options.

II. Elements dependent upon the test constructor or purpose

A: Intent consideration strategy

1. Interpret and answer questions in view of previous idiosyncratic emphases of the test constructor or in view of the test purpose.
2. Answer items as the test constructor intended.
3. Adopt the level of sophistication that is expected.
4. Consider the relevance of specific detail.

B: Cue-using strategy

1. Recognize and make use of any consistent idiosyncrasies of the test constructor which distinguish the correct answer from incorrect options.
2. Consider the relevancy of specific detail when answering a given item.
3. Recognize and make use of specific determiners.
4. Recognize and make use of resemblances between the options and an aspect of the stem.
5. Consider the subject matter and difficulty of neighboring items when interpreting and answering a given item.

Research status on test-wiseness relating to thematic knowledge

The current state of research on test-wiseness shows divergent knowledge and findings about the independence of answering strategies, especially in English-speaking countries. The studies listed below focus on the use of training programs in relation to the application of test-wiseness strategies in multiple-choice tasks. Samson (1985) demonstrated the effectiveness of long-term training programs with pupils in primary and secondary school. In addition, no significant differences could be found in terms of dependency or independence from thematic knowledge in supporting training programs with instructions on test-wiseness. In contrast, Dolly & Williams (1986) proved that cognitive strategies for increasing test-wiseness can be taught to and generalized by undergraduate students so that they improve their test scores. Test-wiseness in combination with different subjects such as History (Thoma & Köller 2018) or Social Studies and Chemistry (Rogers & Tomkowicz 2005) was also investigated. The latter proved that that there is no connection between test-wiseness and thematic knowledge of test persons. However, in contrast to these results, the findings of the study of Thoma & Köller (2018) show the dependency of test-wiseness on the thematic knowledge of individuals. Test participants having thematic knowledge were able to correctly answer and solve a higher number of test-wiseness multiple-choice tasks than participants who did not have thematic knowledge. In addition, it was demonstrated that the effect of training is stronger than that of thematic knowledge. Furthermore, the test-wiseness strategies based on second category of Millman et al. (1965), like the cue-using strategy (II.B), presented the test persons with a greater challenge than multiple-choice tasks based on the first category, like the deductive reasoning strategy (I.D) (Thoma & Köller 2018).
In summary, the current state of research shows divergences. Training programs that specifically target the application and development of (cognitive) strategies increase test-wiseness (Dolly & Williams 1986). By using Test-wiseness strategies, a higher number of multiple-choice tasks can be solved either depending on the thematic knowledge (Thoma & Köller 2018) or independent of thematic knowledge (Rogers & Tomkowicz, 2005).

Mathematical Kangaroo

In 2019, more than 120,000 pupils from around 1,000 Austrian schools took part in the Mathematical Kangaroo in Austria. Across the world, there have been around 6 million competition participants annually in recent years (according to the homepage of the Mathematical Kangaroo - Germany).
The competition currently takes place in six categories: Pre-Ecolier, Ecolier, Benjamin, Cadet, Junior and Student. Each category includes two grades, starting with Pre-Ecolier (1st and 2nd grade). In the category Student, a possible 13th grade is also included. The total number of assignments increases from 15 in the Pre-Ecolier category to 24 in the Ecolier and Benjamin categories to 30 in the Cadet, Junior and Student categories. The processing time is 60 minutes or 75 minutes in the three categories Cadet, Junior and Student. There are three, four and five-point tasks in each category. The difficulty rises with the number of points that can be achieved.

The tasks are sorted by their value. The 3-point tasks are designed in such a way that they should also be able to be solved by the weakest pupils in time. The 4-point tasks are of medium difficulty and the 5-point tasks are intended for specialists (R. Geretschläger, personal communication, June 5, 2018).
As in most international popular competitions, the Mathematical Kangaroo uses a multiple-choice format. The format is 1 out of 5, thus exactly one of five answer options is correct (single-choice format). For each of the five answer options offered, one of the associated letters A, B, C, D or E must be selected and transferred to a pre-defined grid on the cover sheet. If an incorrect answer is entered, a quarter of the points that can be achieved for the correct answer are deducted. Pure guessing of an incorrect selection option is thus sanctioned. Since the participants already start the competition with a credit of a quarter of the total number of points, no negative score can be achieved. When creating tasks for the competition, different levels of mathematical talent are considered. The tasks are created by an international committee of experts. A strategic approach is possible with the present test format and is definitely also intended by the item writers. When designing tasks, traps might be set for the higher-value tasks by using misleading distractors. On the other hand, using the answer options is sometimes necessary for solving the task (R. Geretschläger, personal communication, June 5, 2018).

Test-wiseness at the Mathematical Kangaroo

The principles I.A and I.B (see Table 1) relate to the handling of the entire set of tasks and the competitive situation. In addition to systematic work, due to the limited time for the great amount of tasks, good time management is advantageous. Furthermore, working effectively includes the strategies mentioned above such as deductive reasoning and guessing. Error-avoiding is not only of great importance when solving the tasks, but also when transferring the answers to the solution grid. Interpreting the guiding strategies of I.D in terms of the Mathematical Kangaroo, gives the focus of excluding answers. It is occasionally possible to come closer to the solution by excluding contradicting answer options (see Table 1, I.D.3). Due to the single-choice-format, strategies I.D.2 and I.D.4 cannot be used for the Mathematical Kangaroo (see Table 1).
Deductive reasoning strategies can often be combined with the guessing strategies described in I.C (see Table 1). Since a quarter of achievable points p for the task are subtracted for a wrong answer at the Mathematical Kangaroo, a positive expected value of 1/16 of p is achieved when only one of the five possible answers is excluded. In contrast, pure guessing is not useful, because in this case the expected value is zero.
The guiding strategies of principle II.A (see Table 1) include, for example, recognizing a test purpose or the intention of the task creator. In the Mathematical Kangaroo, it can be assumed that tasks with more points will require more thinking and that answer options that seem too easy may be traps. Traps are included into the answer options via misleading distractors.
Most of the guiding strategies mentioned in II.B (see Table 1), are not directly

transferable to the Mathematical Kangaroo, since, for example, the longest answer is not often the right one. However, experience gained by previous confrontation with the format of the competition may help a lot at the competition. By practicing these types of tasks as well as the mathematical content again and again one can become familiar with them and processing strategies can also be developed from it.

Research question

As a result of the previously treated theories, the question arises, which connection between the principles of test-wiseness and the Mathematical Kangaroo exists. The participants of the competition will be described regarding to their explicit and implicit preparations. Preparations for the competition are distinguished between implicit (extracurricular mathematical activities, e.g. by participating in the Mathematical Olympiads) and explicit (specific preparations by solving prior tasks of the Mathematical Kangaroo) preparation. In addition to this descriptive investigation, the aim is to identify the importance of the strategies for the Austrian winners of the Mathematical Kangaroo within the principles. These research aims lead to the key research question of this study:
"Which test-wiseness based strategies are most important for the Austrian winners of the Mathematical Kangaroo?"

Method

Sample

78 pupils, from grade 3 to 13, who had been honored for their achievements as winners of the Mathematical Kangaroo 2018, were available for data collection. Winners at the regional Styrian awards ceremony as well as the pupils at the national awards ceremony in Vienna were questioned (Styrian students who participated in both ceremonies were only questioned once). Pupils of grades 1 and 2 were not questioned due to the language requirements of the questionnaire. The detailed composition of the sample is listed in Table 2.

Table 2: Composition of the sample

Number of participants	valid	73
	invalid	5
Gender	Female	19
	Male	54
Age	Young (grade 3-8)	47
	Old (grade 9-13)	26
Ceremony	regional (Styria)	41
	national (Vienna)	37

Questionnaire

Data was collected with the KATS-questionnaire, that has been developed within the studies for the employment and usage of strategies by answering 42 items. The construction of the questionnaire at the content level bases on the Taxonomy of test-wiseness principles by Millman et al. (1965). The principles and strategies taken from literature were adapted to the Mathematical Kangaroo and formulated clearly and in a manner that is appropriate to pupils. This procedure gave 38 items, which were integrated into KATS. The remaining items concern the preparation and necessary personal data (e.g. grade). Below, these 38 items are assigned to the principles of test-wiseness:

Table 3: Composition of the KATS-questionnaire

Test-wiseness principles	Number of items
Time-using strategy	6
Error-avoidance strategy	5
Guessing strategy	12
Deductive reasoning strategy	3
Intent consideration strategy	7
Cue-using strategy	5

Prior to conducting the survey, draft versions of the questionnaire were forwarded to experienced former winners and also some task creators of the competition, to gather feedback regarding the comprehensibility and the question of whether all aspects of the competition were addressed. At the content level, items were constructed allowing assignable conclusions concerning ”elements independent of test constructor or purpose” and on the other hand also allowing conclusions concerning ”elements dependent of test constructor or purpose”. The articulated statements were rated by the students on a five-point Likert scale (agree (1), rather agree (2), partly agree (3), rather disagree (4), strongly disagree (5)).

Data analysis

After the collection of data using the KATS-questionnaire, the information was processed in Excel and SPSS 23. The ranking within the principles of test-wiseness was performed by comparison of the arithmetic means, also taking the standard deviation into account. This rests on the use of a five-point Likert scale with equidistance between the answer options. Due to statistical robustness, the medians and the interquartile ranges were additionally determined and specified in the section of results.

Ethics

The data were collected in May and June 2018 at the Styrian Mathematical Kangaroo award ceremony in Graz and at the Austrian award ceremony in Vienna.

Declarations of consent were obtained during the ceremonies from the legal guardians of the pupils for participating in the survey. At this time, the research project was briefly explained, a contact address (email) was given, the confidential handling of the collected data was assured and the voluntary nature of the students' participation was pointed out. If a guardian was not present, an email address was requested so that they were able to give their consent afterwards. A total of 21 girls and 57 boys were interviewed. In addition to the items concerning test-wiseness, additional data (name, gender, school level, previous participation in the competition and at the award ceremony, Math grade and attendance at Math events / courses) were also collected. The name was only required to assign the declarations of consent. The data were evaluated using raw data and anonymized when the results were written.

Results

Preparation for the competition

The Austrian winners were categorised with respect to their preparation for the competition. 20 winners prepared only explicitly, and 17 winners prepared implicitly. 17 winners prepared both explicitly and implicitly for the competition and the remaining 19 winners did not prepare at all. They were furthermore split by gender and age (see the following table).

Table 4: Division of the winners regarding preparation

Type of preparation	Age (young/old)	Gender (Female/Male)	Total
Explicit	18/2	8/12	20
Implicit	6/11	4/13	17
Implicit and explicit	8/9	4/13	17
No preparation	15/4	3/16	19
Total	47/26	19/54	73

Top ranked strategies

Table 5 below shows the principles of test-wiseness (I.A, I.B, I.C, I.D, II.A and II.B) and (the English translation of) selected strategies of the KATS-questionnaire, which were derived from theory. Within the principles (e.g. Time-using strategy), strategies are ranked by their importance for the Austrian winners of the Mathematical Kangaroo (see Rank). This ranking is based on comparison of means (arithmetic mean AM) taking account of the standard deviation (SD). Further entries are the median (MD) and the interquartile range (IQR). In Table 5, two strategies per principle have been included, with a third strategy added if the second place could not have been clearly identified otherwise.

Index	Rank		AM	SD	MD	IQR
		Time-using strategy				
I.A#1	1	In the final minutes I try to attack unsolved tasks rather than checking everything again.	2.04	1.18	2	2
I.A#2a	2	In a first run, I work on all tasks that I can solve quickly.	2.83	1.39	3	2
I.A#2b	3	I mark unsolved tasks so that I can work on them again later.	2.86	1.80	2	4
		Error-avoidance strategy				
I.B#1	1	I take time to carefully read the tasks.	1.60	0.80	1	1
I.B#2	2	I check again whether the answer entered on the cover sheet actually match with the received answers.	1.71	1.29	1	1
		Guessing strategy				
I.C#1	1	I just mark those answer fields on the cover sheet that I am sure of.	2.56	1.35	2	3
I.C#2	2	I try to fill in all answer fields on the cover sheet, as otherwise the chance of winning is a priori too small.	2.94	1.57	3	3.25
		Deductive reasoning strategy				
I.D#1	1	I look at the answer options before starting to calculate and try to first rule out obviously incorrect answer options.	2.42	1.35	2	2
I.D#2	2	If I don‘t see immediately how I can solve a task, I try to find a solution using the answer options.	2.50	1.31	2	2
		Intent consideration strategy				
II.A#1	1	I solve the 3-point tasks first.	1.35	0.99	1	0
II.A#2a	2	I check 4- and/or 5-point tasks for possible traps.	1.96	1.27	2	1
II.A#2b	2	I don‘t skip 3-point tasks.	1.96	1.26	1	1.75
		Cue-using strategy				
II.B#1	1	I practice former tasks of the Mathematical Kangaroo before the competition.	3.33	1.66	4	3
II.B#2	2	Before the competition I was told in Math lessons about the best way to solve the tasks.	4.10	1.42	5	1.5

Table 5: Top-rank strategies

Among the four principles of type I (elements independent of test constructor or purpose), the top-ranked strategies belonging to the two principles time using-strategy (I.A) and error-avoidance strategy (I.B) have by far lower means than the top-strategies belonging to the principles guessing strategy (I.C) or deductive reasoning strategy (I.D). Considering the top-ranked strategies of principles of the second type (elements dependent of test constructor or purpose), the strategies belonging to the Intent consideration strategy (II A) have by far lower means than those belonging to the cue-using strategy (II.B).

The aspects of test-wiseness at the competition that were given lower priority will be taken up again in the next section.

Discussion

In this chapter, the composition of the Austrian winners with respect to their preparation is first examined. The principles of Millman et al. (1965) are then discussed on the basis of the Austrian winners' answer to the KATS-questionnaire. Subsequently, the top ranked strategies within the principles are compared to the less important strategies from the Austrian winners' point of view. Finally, an outlook on the further development of KATS is given and research questions are asked for further investigations.

Composition of the winners in terms of preparation, age and gender

The groups of explicit and implicit preparation correspond to the groups of explicit preparation and prior knowledge differentiated by Thoma & Köller (2018), since an additional extra-curricular occupation with mathematical topics automatically leads to further training of the competition's mathematical content. Based on Thoma & Köller (2018), explicit preparation describes the test-wiseness training and implicit preparation describes the thematic knowledge.
Table 4 shows that the Austrian winners are distributed almost equally across the four types of preparation. However, differences in the composition of each of the four groups in terms of age and gender are very noticeable. It is striking that younger winners prepare just explicitly far more often than older winners. The implicit preparation shows the opposite picture in relation to age. While a third of the older winners prepare themselves explicitly and implicitly and only a few of this group do not prepare at all, a third of the younger winners do not prepare for the competition at all. This could be due to the lack of opportunities for extracurricular support in mathematics. For example, greater focus on the Mathematical Olympiad and elective subjects with mathematical content is available for older students. The Mathematical Kangaroo, designed as a popular competition, tries to provide something for all ages. This is obviously accepted by the younger winners, since earlier competition tasks serve to explicitly prepare for future competitions.
The number of female winners is far smaller than the number of male winners, even though the proportion of participants in the competition in Austria 2018 was almost the same (according to the homepage of the Kangaroo of Mathematics - Austria, the ratio of girls to boys was 1: 1.06). This aspect has also been noticed in all statistical surveys of the organizers of the competition in recent years (Mathematical Kangaroo, online). The reasons for this cannot be examined in more detail here due to a different focus of the data collection. What is striking is the fact that almost a third of all male winners do not prepare for the competition, whereas almost all female Austrian winners do prepare for the competition in some way.
8.2 Winners evaluate the principles of test-wiseness within the Mathematical Kangaroo The principles of Millman et al (1965), described in Chapter 1, served as the basis for the development and formulation of the items in the KATS-questionnaire. The main task here was to adapt the principles to the Mathematical Kangaroo. The strategies (items) of KATS were always derived according to Millman et al (1965). It should be noted that, due to the format 1 out of 5, some of the key strategies of

the deductive reasoning strategy do not apply to the Mathematical Kangaroo.
As a next step, these strategies were evaluated by Austrian winners. It was noticed here that some principles are more important for the winners than others, which is shown by smaller means in the top-ranked strategies. It is particularly noticeable that the principle of error avoidance strategy has been given much greater importance than the principle of guessing strategy. This could be due to the fact that some winners did not have to guess and that the willingness to take risks due to penalty points is low despite positive expected value. There can also be several reasons for the different importance of the six principles when answering the questionnaire: the derivation of strategies based on theory from the competition, the conception of the Mathematical Kangaroo itself or the special composition of the sample.
8.3 Comparing top-ranked strategies and less-important strategies in the winner‘s perspective The strategies (items) of KATS split each principle into the groups of top-ranked strategies and less important strategies (li-s). The label li-s refers to all those strategies to which the winners assigned a lesser role in contrast to the top-ranked strategies. Table 5 contains only the top-ranked strategies (I.A#1) to (II.B#2). In the following, top-ranked strategies and less important strategies are contrasted.
Within the principle of time-using, the winners prefer to work on tasks which they can solve quickly first (I.A#2a) instead of getting an overview of all tasks (li-s) and trying to attack unsolved tasks in the last few minutes of the competition (I.A#1) instead of checking their answers (li-s). Surprisingly, (I.A#1) is a top ranked strategy, a strategy not effective according to Millman et. al (1965), see I.A.5 in Table 1.
Looking at the principle error-avoidance, winners tend to carefully read the tasks (I.B#2) and try to avoid errors during the transfer of their answers to the cover sheet (I.B#2) rather than checking their approaches again (li-s).
The top-ranked strategies within the principle of guessing seem to be contradictory. Whereas the winners just answer those tasks, which they are sure of (I.C#1), they simultaneously try to fill in all answer fields in order to win (I.C#2). This contradiction may be due to the fact that the winners assume that they are able to solve all (or at least almost all) tasks without having to guess. The top-ranked strategies involve a lower risk than the less-important strategies. For instance, comparing to the top-ranked strategies, the winners prefer not to guess, even if they can exclude three of the five answer options (li-s). A possible conclusion is that the winners avoid guessing or risk. All strategies based on the principle of deductive reasoning, refer to answer-option-based working backwards, which is described by Andritsch et. al (2020). This explains the similar means of (I.D#1) and (I.D#2).
Within the principle of intent consideration, winners tend to solve tasks first, which are expected to be easy (II.A#1) and not skipping them (II.A#2b). The order of attacking the tasks seem to be independent of the content (Rogers & Tomkowicz 2005) of the tasks (li-s). Furthermore, especially more valuable tasks are checked for possible traps (II.A#2a) instead of checking 3-point tasks for traps as well (li-s). These findings go hand in hand with the conception of the competition, since on the one hand solving all 3-point tasks is essential for a good performance and on the other hand the 3-point tasks usually do not involve any traps.

Practising for the competition by solving tasks from previous competitions (II.B#1) seems to be more important for the winners than other strategies within the principle of cue-using, such as considering mathematical properties of the current year (li-s). The average of 4.1 at the top-ranked strategy (II.B#2) shows that the Mathematical Kangaroo is hardly addressed within the winner's regular Math classes and that the principle of cue-using is just a minor aspect for them at the Mathematical Kangaroo.

Conclusion and Outlook

The importance of test-wiseness at the Mathematical Kangaroo can be visualized by means of the previously formulated strategies and discussion. For this sample, consisting of Austrian winners, top-strategies have emerged within the six principles. These could also be of great importance for other multiple-choice competitions with mathematical content. Relevant for practical application could also be the fact that approximately half of the winners prepared for the competition explicitly, by means of training previous tasks of the Mathematical Kangaroo. Based on this pilot study the authors are going to carry out a general survey of participants of the Mathematical Kangaroo and/or comparable mathematical competitions. An interesting question might be, if some of the top-ranked strategies of the Austrian winners differ from strategies used by the other participants. Acknowledgements: The authors would like to thank Evita Hauke and Robert Geretschläger for comments on earlier versions, fruitful discussions and general support during the work on this paper.

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The 61st International Mathematical Olympiad

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The 61st International Mathematical Olympiad (IMO) was held 19–28 September 2020. Due to COVID19, this was a distributed IMO administered from St Petersburg, Russian Federation.

This was the second time that Russia has hosted the IMO. A total of 616 high school students from 105 countries participated. Of these, 56 were female.

The advent of COVID19 seriously jeopardised the viability of running the IMO this year. However, the organisers of IMO 2020 and the IMO Board decided that it was "vital to provide a fully official IMO 2020 in September to all the young mathematicians who have been preparing to compete for many years. In order to do this remotely, a completely new virtual IMO format has been invented, with security protocols in place so that everyone can have full confidence in the integrity of the results."

To ensure the integrity of the contest, students sat the contest papers at Exam Centres in their own countries. The exams were invigilated from St Petersburg using the *Zoom* video conferencing platform. Moreover, the IMO Board appointed an *IMO Commissioner* for each Exam Centre. The Commissioner was a resident of the country, but generally not a citizen. They were to be trusted individuals at each Exam Centre who would ensure fair play in the administering of the contest and in the scanning and uploading of students scripts.

As per normal IMO rules, each participating country may enter a team of up to six students, a Team Leader and a Deputy Team Leader.[1]

Participating countries also submit problem proposals for the IMO. This year there were 149 problem proposals from 39 countries. The local Problem Selection Commit-

[1]The IMO regulations also permit countries to enter a small number of additional staff as Observers. These may fulfil various roles such as meeting child safety obligations, assisting with marking and coordination or learning about how to host an IMO.

tee shortlisted 32 of these for the contest and then went on to set the two competition papers.

At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. The Jury normally makes various decisions such as approving marking schemes, and setting medal boundaries. However, this year, decisions such as these were made on behalf of the Jury by the Jury Chair taking advice from the IMO Board and, if necessary, the IMO Ethics Committee.

The six problems that ultimately appeared on the IMO contest papers may be described as follows.

1. An easy classical geometry problem proposed by Poland.
2. A medium weak algebraic inequality proposed by Belgium.
3. A difficult combinatorics problem proposed by Hungary. A creative part of this problem was to realise that it could be translated into a graph theory problem.
4. An easy graph theory problem proposed by India.
5. A medium to easy number theory problem with a dash of combinatorial thinking proposed by Estonia.
6. A very difficult problem in combinatorial geometry proposed by Taiwan.

These six problems were posed in two exam papers held on 21 September and 22 September for 4.5 hours each day starting at a time between 07:30 and 12:00 Universal Coordinated Time (UTC). This helped ensure the integrity of the contest as no student would finish the contest before another had started. This led to some uncomfortable situations such the New Zealand contestants finishing the exams at midnight while some students in the Americas would begin the exams at 7am in the morning.

Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

After the exams, the Leaders and their Deputies spent about two to three days assessing the work of the students from their own countries, guided by marking schemes. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores.

The contestants found Problem 1 to be the easiest with an average score of 5.31. Problem 6 was the hardest, averaging just 0.28. Only four contestants scored full marks on this very challenging problem.[2] The score distributions by problem number were as follows.

Mark	P1	P2	P3	P4	P5	P6
0	117	291	465	213	294	481
1	26	29	47	11	83	126
2	5	129	3	3	0	1
3	5	9	14	42	2	1
4	2	4	40	35	1	1
5	3	7	0	14	9	1
6	7	9	5	13	4	1
7	451	138	42	285	223	4
Mean	5.31	2.25	0.94	3.94	2.80	0.28

The medal cuts were set at 31 points for Gold, 24 for Silver and 15 for Bronze. The medal distributions[3] were as follows.

	Gold	Silver	Bronze	Total
Number	49	112	155	316
Proportion	8.0%	18.2%	25.2%	51.3%

These awards were presented at the closing ceremony. Of those who did not get a medal, a further 173 contestants received an Honourable Mention for scoring full marks on at least one problem.

Jinmin Li of the People's Republic of China was the sole contestant who achieved the most excellent feat of a perfect score of 42.

The 2020 IMO was organised by the Ministry of Education of the Russian Federation, Herzen State Pedagogical University of Russia, the Government of St. Petersburg, and the Presidential Physics and Mathematics Lyceum No. 239.

Hosts for future IMOs have been secured as follows.

14–24 July, 2021	St. Petersburg, Russian Federation
6–16 July, 2022	Oslo, Norway
2–13 July, 2023	Chiba, Japan
2024	vacant
2025	Melbourne, Australia

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org

[2]Only three other IMO problems have had a smaller number of complete solutions in the modern IMO era, that is, since 1983 when the format of the IMO stabilised. They were problem 3 in 2007 (two solutions), problem 6 in 2009 (three solutions), and problem 3 in 2017 (two solutions).

[3]The total number of medals must (normally) be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals should be approximately in the ratio 1:2:3.

Monday, 21. September 2020

Problem 1. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 2. The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

Problem 3. There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each colour.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

IMO 2020

Tuesday, 22. September 2020

Problem 4. There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 5. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.

For which n does it follow that the numbers on the cards are all equal?

Problem 6. Prove that there exists a positive constant c such that the following statement is true:

> Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ separates a set of points $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

Note. Weaker results with $cn^{-1/3}$ replaced by $cn^{-\alpha}$ may be awarded points depending on the value of the constant $\alpha > 1/3$.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Some Country Totals

Rank	Country	Total
1	People's Republic of China	215
2	Russian Federation	185
3	United States of America	183
4	Republic of Korea	175
5	Thailand	174
6	Italy	171
6	Poland	171
8	Australia	168
9	United Kingdom	167
10	Brazil	165
11	Ukraine	164
12	Canada	161
13	Hungary	160
14	France	154
15	Romania	152
16	Singapore	151
17	Vietnam	150
18	Georgia	149
18	Islamic Republic of Iran	149
18	Japan	149
21	Israel	146
21	Kazakhstan	146
23	Czech Republic	145
23	Taiwan	145
25	Serbia	144
26	Germany	140
26	Turkey	140
28	Hong Kong	139
29	Mongolia	135
29	Netherlands	135

Distribution of Awards at the 2019 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	40	0	0	1	2
Algeria	5	0	0	0	0
Argentina	99	1	0	2	2
Armenia	126	0	2	3	1
Australia	168	2	3	1	0
Austria	66	0	1	1	3
Azerbaijan	66	0	0	1	5
Bangladesh	118	0	1	5	0
Belarus	111	0	0	6	0
Belgium	99	0	1	3	1
Bolivia	21	0	0	0	2
Bosnia and Herzegovina	85	0	0	5	1
Botswana	5	0	0	0	0
Brazil	165	1	5	0	0
Bulgaria	118	0	1	3	2
Canada	161	3	1	2	0
Chile	19	0	0	0	1
Colombia	129	0	2	3	1
Costa Rica	28	0	0	0	2
Croatia	130	0	2	3	1
Cyprus	36	0	0	0	3
Czech Republic	145	1	2	3	0
Denmark	95	0	0	3	3
Ecuador	71	0	0	1	5
El Salvador	61	0	0	1	3
Estonia	84	0	1	2	3
Finland	81	0	1	1	3
France	154	1	3	1	1
Georgia	149	1	2	2	1
Germany	140	0	3	3	0
Ghana	21	0	0	1	0
Greece	86	0	0	4	2
Honduras	28	0	0	0	3
Hong Kong	139	0	3	3	0
Hungary	160	3	0	3	0
Iceland	22	0	0	0	1
Indonesia	130	2	0	2	2

Country	Total	Gold	Silver	Bronze	HM
Iraq	15	0	0	0	1
Ireland	53	0	0	0	5
Islamic Republic of Iran	149	1	3	2	0
Israel	146	1	2	3	0
Italy	171	2	3	1	0
Japan	149	0	5	1	0
Kazakhstan	146	0	3	2	1
Kenya	2	0	0	0	0
Kosovo	38	0	0	0	5
Kyrgyzstan	42	0	0	1	2
Latvia	64	0	0	0	6
Lithuania	94	0	0	3	3
Luxembourg	3	0	0	0	0
Macau	76	0	0	2	3
Malaysia	130	1	2	2	1
Mexico	111	1	0	4	1
Mongolia	135	1	2	1	2
Montenegro	50	0	1	0	3
Morocco	26	0	0	0	2
Myanmar	23	0	0	0	3
Nepal	4	0	0	0	0
Netherlands	135	2	1	3	0
New Zealand	102	1	0	2	3
Nicaragua	19	0	0	0	2
Nigeria	30	0	0	0	2
North Macedonia	51	0	0	0	5
Norway	83	1	0	1	3
Oman	2	0	0	0	0
Pakistan	34	0	0	0	3
Panama	32	0	0	1	1
Paraguay	44	0	0	1	3
People's Republic of China	215	5	1	0	0
Peru	127	0	3	2	0
Philippines	113	1	0	2	3
Poland	171	2	3	1	0
Portugal	112	0	1	3	2
Puerto Rico	38	0	0	1	3
Republic of Korea	175	2	3	1	0
Republic of Moldova	97	0	2	2	1

Country	Total	Gold	Silver	Bronze	HM
Romania	152	1	2	3	0
Russian Federation	185	2	4	0	0
Saudi Arabia	82	0	0	2	4
Serbia	144	1	2	2	1
Singapore	151	0	4	2	0
Slovakia	117	0	2	4	0
Slovenia	117	1	1	1	3
South Africa	77	0	0	3	3
Spain	133	0	2	4	0
Sri Lanka	77	0	0	3	3
Sweden	117	0	1	4	1
Switzerland	100	0	0	4	1
Syria	102	0	2	1	3
Taiwan	145	0	3	3	0
Tajikistan	47	0	0	0	5
Tanzania	14	0	0	0	1
Thailand	174	2	3	1	0
Trinidad and Tobago	13	0	0	0	0
Tunisia	52	0	1	0	3
Turkey	140	0	2	4	0
Turkmenistan	31	0	0	0	4
Uganda	29	0	0	0	3
Ukraine	164	0	6	0	0
United Kingdom	167	1	4	1	0
United States of America	183	3	3	0	0
Uruguay	36	0	0	0	4
Uzbekistan	60	0	0	0	6
Venezuela	33	0	0	0	4
Vietnam	150	2	1	2	1
Total (105 teams, 616 contestants)		**49**	**112**	**155**	**173**

N.B. Not all countries entered a full team of six students.

Angelo Di Pasquale
IMO Team Leader, Australia

International Mathematics Tournament of the Towns

Andy Liu

Andy Liu is a Canadian mathematician. He is a professor emeritus in the Department of Mathematical and Statistical Sciences at the University of Alberta. Liu attended New Method College in Hong Kong.He then did his undergraduate studies in mathematics at McGill University, and earned his Ph.D. in 1976 from the University of Alberta, under the supervision of Harvey Abbott, with a dissertation about hypergraphs. He was the leader of the Canadian team at the International Mathematical Olympiad in 2000 (South Korea) and 2003 (Japan) and acts as vice-president of the Tournament of Towns.

Selected Problems with Solutions from the Spring 2020 Papers

1. Does there exist a positive multiple of 2020 which contains each of the ten digits the same number of times?

   Solution:
   Note that $2020 = 2^2 \times 5 \times 101$. If the last two digits of a number are 20, it will be a multiple of $2^2 \times 5$. The number 19193838474756562020 is a multiple of 2020, and contains each of the digits exactly twice. An example which contains each of the digits exactly once is 1237548960.

2. A dragon has 41! heads and the knight has three swords. The gold sword cuts off half of the current number of heads of the dragon plus one more. The silver sword cuts off one third of the current number of heads plus two more. The bronze sword he cuts off one fourth of the current number of heads plus three more. The knight can can use any of the three swords in any order. However, if the current number of heads of the dragon is not a multiple of 2 or 3, the swords do not work and the dragon eats the knight. Will the knight be able to kill the dragon by cutting off all its heads?

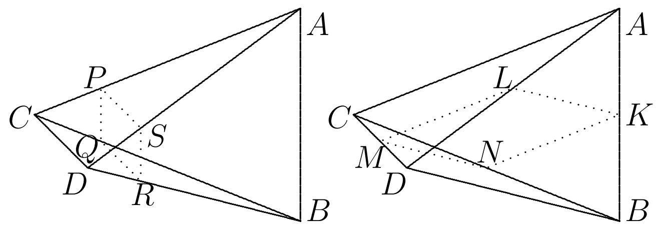
   Solution:
   If the number of heads is even, the knight can reduce the number of heads again to even number. Indeed, if the number of heads is $4n - 2$, then after gold sword strike it will become $2n - 2$. If the number of heads is $4n$, then after bronze sword strike it will become $3n - 3$, and after the next silver sword strike it become $2n - 4$. The knight can act this way until there are four or two heads left. The task is completed with one final strike from the bronze or the gold sword, respectively.

3. What is the minimum number of points on the surface of a sphere such that for every hemisphere except one, at least one of the points lies in the interior of the hemisphere?

Solution:
Suppose we have only three points. They determine a circle which divides the sphere into two parts. If both parts are hemispheres, then neither contains any of these three points in its interior. Otherwise, one part contains a hemisphere whose equator is on a plane parallel to the plane containing the three points. It does not contain any of these three points in its interior. We now show that four points are sufficient. Let N be the North Pole and let ABC be an equilateral triangle inscribed in the Equator. Then the Southern Hemisphere does not contain any of A, B, C and N in its interior. For any other hemisphere, its equator must intersect the Equator in two antipodal points. Hence it must contain in its interior one half of the Equator, and hence at least one of the points A, B and C.

4. Is it possible for a tetrahedron to have two cross-sections, one a square of side length at most 1 and the other a square of side length at least 100?

Solution:
Let the vertices of the small square cross-section be $P(0, \frac{1}{3}, \frac{1}{3})$, $Q(0, \frac{1}{3}, -\frac{1}{3})$, $R(0, -\frac{1}{3}, -\frac{1}{3})$ and $S(0, -\frac{1}{3}, \frac{1}{3})$. Let two of the vertices of the tetrahedron be $A(\ell, 0, h)$ and $B(\ell, 0, -h)$. The other two vertices are the point C of intersection of the extensions of AP and BQ, along with the point D of intersection of the extensions of AS and BR. Note that $AC = AD = BC = BD > 200$ with a suitable choice of ℓ, and AC is perpendicular to BD with a suitable choice of h. Let K, L, M and N be the respective midpoints of AB, AD, CD and BC. Then $KL = MN = \frac{1}{2}BC > 100$. Moreover, KL and MN are parallel to BC. Hence $KLMN$ is a parallelogram. Since $AC = BD$ and they are perpendicular to each other, $KLMN$ is the desired large square cross-section.



5. On each of $2n$ children is placed a black hat or a white hat. There are n hats of each color. The children form one or several dancing circles, with at least two children in each, and the colors of the hats alternate within each circle. Prove that this can be done in exactly $(2n)!$ different ways.

Solution:
There are $\binom{2n}{n}$ ways of placing the hats on the children. We claim that the number of ways of distributing them into dancing circles of even length is $(n!)^2$. We use induction on n. For $n = 1$, the number of ways is $1 = (1!)^2$. Suppose the result holds up to some $n \geq 1$. Consider the next case with $2(n + 1)$ children and focus on a particular child wearing a white hat. This child is in a dancing circle of length $2k$ where $1 \leq k \leq n+1$. We can choose the remaining $k - 1$ children in the dancing circle wearing white hats in $\binom{n}{k-1}$ ways, choose

the k children in the dancing circle wearing black hats in $\binom{n+1}{k}$ ways and form the dancing circle in $(k-1)!k!$ ways. By the induction hypothesis, the other dancing circles may be formed in $((n-k+1)!)^2$ ways. The number of ways is $\binom{n}{k-1}\binom{n+1}{k}(k-1)!k!((n-k+1)!)^2 = n!(n+1)!$, which is independent of k. Summing from $k=1$ to $k=n+1$, we have $((n+1)!)^2$, completing the inductive argument. The desired result follows immediately.

6. For which integers $n \geq 2$ is it possible to write real numbers into the squares of an $n \times n$ table, so that every integer from 1 to $2n(n-1)$ appears exactly once as the sum of the numbers in two adjacent squares?

Solution:
This is possible for all n. Consider first the case $n = 2k$. The numbers in the first column start with 0, and increase by $2k$ and $2k-1$ alternatingly, resulting in 0, $2k$, $4k-1$, $6k-1$, $8k-2$, …, $(4k-4)k-(k-1)$ and $(4k-2)k-(k-1)$. In odd-numbered rows, the numbers increase by 1 and 0 alternatingly. In even-numbered rows, they increase by 0 and 1 alternatingly. Thus the numbers in the last column are k, $3k-1$, $5k-1$, $7k-2$, $9k-2$, …, $(4k-3)k-(k-1)$ and $(4k-1)k-k$. The following table illustrates the construction for $k = 3$.

0	1	1	2	2	3
6	6	7	7	8	8
11	12	12	13	13	14
17	17	18	18	19	19
22	23	23	24	24	25
28	28	29	29	30	30

The pairwise sums along the first row are 1, 2, …, $2k-1$. The pairwise sums between the first row and the second row are $2k$, $2k+1$, …, $4k-1$. The pairwise sums along the second row are $4k$, $4k+1$, …, $6k-2$. The pairwise sums between the second row and the third row are $6k-1$, $6k$, …, $8k-2$, and so on. The pairwise sums along the last row are $2(4k-2)k-(k-1)) = 8k^2-5k+1$, $8k^2-5k+2$, …, $8k^2-4k = 2n(n-1)$.
The construction for $n = 2k+1$ is similar. The numbers in the first column start with 0, and increase by $2k+1$ and $2k$ alternatingly, resulting in 0, $2k+1$, $4k+1$, $6k+2$, $8k+2$, …, $(4k-2)k+k$ and $(4k)k+k$. In odd-numbered rows, the numbers increase by 1 and 0 alternatingly. In even-numbered rows, they increase by 0 and 1 alternatingly. Thus the numbers in the last column are k, $3k+1$, $5k+1$, $7k+2$, $9k+2$, …, $(4k-1)k+k$ and $(4k+1)k+k$. The largest pairwise sum is indeed $2((4k+1)k+k) = 8k^2+4k = 2n(n-1)$. The following table illustrates the construction for $k = 3$.

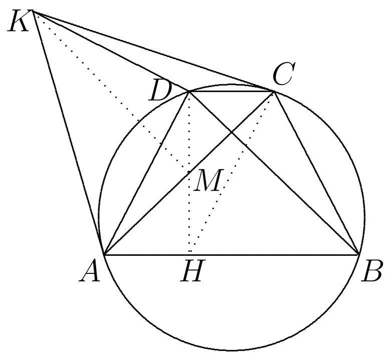
0	1	1	2	2	3	3
7	7	8	8	9	9	10
13	14	14	15	15	16	16
20	20	21	21	22	22	23
26	27	27	28	28	29	29
33	33	34	34	35	35	36
39	40	40	41	41	42	42

7. In $ABCD$, AB is parallel to CD and $AB = 3CD$. The tangents to the circumcircle at A and C intersect at K. Prove that $\angle KDA = 90°$.

Solution:
Let H be the point on AB such that $AH = CD$. Then $AHCD$ is a parallelogram. Its diagonals DH and AC bisect each other at the centre M. Since $KA = KC$, KM is perpendicular to AC. Since $AB = 3CD$, DH is perpendicular to AB. Now

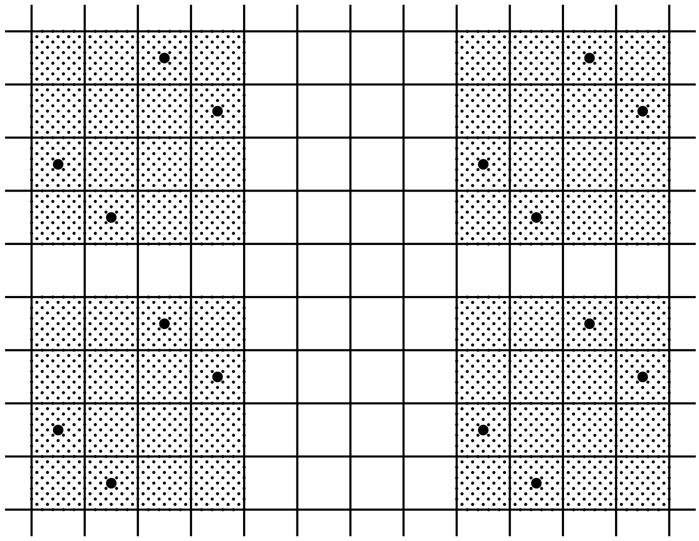
$$\angle KMD = 90° - \angle DMC = 90° - \angle HMA = \angle CAH = \angle DBA = \angle KAD.$$

Hence $AKDM$ is cyclic. It follows that $\angle KDA = \angle KMA = 90°$.



8. For which k is it possible to place a finite number of queens on the squares of an infinite chessboard, so that the number of queens in each row, each column and each diagonal is either 0 or exactly k?

Solution:
This is possible for every positive integer k. The configuration will consists of k^2 groups each with k^2 queens, for a total of k^4 queens. The groups are well algined and properly spaced. For $k = 1$, a lone queen satisfies the condition. The diagram below shows the construction for $k = 2$. The constructions for higher values of k are analogous.



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