6

HARMONIC FUNCTIONS

This chapter is devoted to the Laplace equation. We introduce two of its important properties, the maximum principle and the rotational invariance. Then we solve the equation in series form in rectangles, circles, and related shapes. The case of a circle leads to the beautiful Poisson formula.

6.1 LAPLACE'S EQUATION

If a diffusion or wave process is stationary (independent of time), then $u_t \equiv 0$ and $u_{tt} \equiv 0$. Therefore, both the diffusion and the wave equations reduce to the *Laplace equation*:

$$
\begin{aligned}
u_{xx} &= 0 && \text{in one dimension} \\
\nabla \cdot \nabla u = \Delta u &= u_{xx} + u_{yy} = 0 && \text{in two dimensions} \\
\nabla \cdot \nabla u = \Delta u &= u_{xx} + u_{yy} + u_{zz} = 0 && \text{in three dimensions}
\end{aligned}
$$

A solution of the Laplace equation is called a *harmonic function.*

In *one* dimension, we have simply $u_{xx} = 0$, so the only harmonic functions in one dimension are $u(x) = A + Bx$. But this is so simple that it hardly gives us a clue to what happens in higher dimensions.

The inhomogeneous version of Laplace's equation

$$\boxed{\Delta u = f} \tag{1}$$

with f a given function, is called *Poisson's equation*.

Besides stationary diffusions and waves, some other instances of Laplace's and Poisson's equations include the following.

1. **Electrostatics.** From Maxwell's equations, one has curl $\mathbf{E} = 0$ and div $\mathbf{E} = 4\pi\rho$, where ρ is the charge density. The first equation implies $\mathbf{E} = -\text{grad}\ \phi$ for a scalar function ϕ (called the *electric potential*). Therefore,
$$\Delta\phi = \text{div}(\text{grad}\ \phi) = -\text{div}\ \mathbf{E} = -4\pi\rho,$$
which is Poisson's equation (with $f = -4\pi\rho$).
2. **Steady fluid flow**. Assume that the flow is irrotational (no eddies) so that curl $\mathbf{v} = 0$, where $\mathbf{v} = \mathbf{v}(x, y, z)$ is the velocity at the position (x, y, z), assumed independent of time. Assume that the fluid is incompressible (e.g., water) and that there are no sources or sinks. Then div $\mathbf{v} = 0$. Hence $\mathbf{v} = -\text{grad}\ \phi$ for some ϕ (called the *velocity potential*) and $\Delta\phi = -\text{div}\ \mathbf{v} = 0$, which is Laplace's equation.
3. **Analytic functions of a complex variable.** Write $z = x + iy$ and
$$f(z) = u(z) + iv(z) = u(x + iy) + iv(x + iy),$$
where u and v are real-valued functions. An analytic function is one that is expressible as a power series in z. This means that the powers are not $x^m y^n$ but $z^n = (x + iy)^n$. Thus
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
(a_n complex constants). That is,
$$u(x + iy) + iv(x + iy) = \sum_{n=0}^{\infty} a_n (x + iy)^n.$$
Formal differentiation of this series shows that
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
(see Exercise 1). These are the *Cauchy–Riemann equations*. If we differentiate them, we find that
$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$
so that $\boldsymbol{\Delta u = 0}$. Similarly $\boldsymbol{\Delta v = 0}$, where Δ is the two-dimensional laplacian. Thus the real and imaginary parts of an analytic function are harmonic.
4. **Brownian motion.** Imagine brownian motion in a container D. This means that particles inside D move completely randomly until they hit the boundary, when they stop. Divide the boundary arbitrarily into two pieces, C_1 and C_2 (see Figure 1). Let $u(x, y, z)$ be the probability that a particle that begins at the point (x, y, z) stops at some point of C_1. Then it can be deduced that
$$\boldsymbol{\Delta u = 0} \text{ in } D$$
$$u = 1 \text{ on } C_1 \qquad u = 0 \text{ on } C_2.$$
Thus u is the solution of a Dirichlet problem.

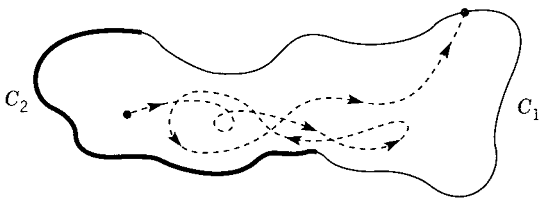

Figure 1

As we discussed in Section 1.4, the basic mathematical problem is to solve Laplace's or Poisson's equation in a given domain D with a condition on bdy D:

$$\boxed{\begin{array}{c} \Delta u = f \text{ in } D \\ u = h \quad \text{or} \quad \dfrac{\partial u}{\partial n} = h \quad \text{or} \quad \dfrac{\partial u}{\partial n} + au = h \quad \text{on bdy } D. \end{array}}$$

In one dimension the only connected domain is an interval $\{a \leq x \leq b\}$. We will see that what is interesting about the two- and three-dimensional cases is the geometry.

MAXIMUM PRINCIPLE

We begin our analysis with the maximum principle, which is easier for Laplace's equation than for the diffusion equation. By an *open set* we mean a set that includes none of its boundary points (see Section A.1).

Maximum Principle. Let D be a connected bounded open set (in *either* two- *or* three-dimensional space). Let *either* $u(x, y)$ *or* $u(x, y, z)$ be a harmonic function in D that is continuous on $\overline{D} = D \cup (\text{bdy } D)$. *Then the maximum and the minimum values of u are attained on* bdy D *and nowhere inside (unless $u \equiv$ constant).*

In other words, a harmonic function is its biggest somewhere on the boundary and its smallest somewhere else on the boundary.

To understand the maximum principle, let us use the vector shorthand $\mathbf{x} = (x, y)$ in two dimensions or $\mathbf{x} = (x, y, z)$ in three dimensions. Also, the radial coordinate is written as $|\mathbf{x}| = (x^2 + y^2)^{1/2}$ or $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2}$. The maximum principle asserts that there exist points $\mathbf{x}_M$ and $\mathbf{x}_m$ *on bdy D* such that

$$u(\mathbf{x}_m) \leq u(\mathbf{x}) \leq u(\mathbf{x}_M) \tag{2}$$

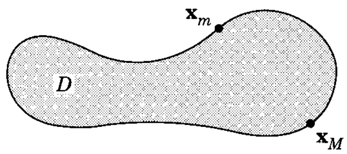

Figure 2

for all $\mathbf{x} \in D$ (see Figure 2). Also, there are no points inside D with this property (unless $u \equiv$ constant). There could be several such points *on* the boundary.

The *idea* of the maximum principle is as follows, in two dimensions, say. At a maximum point inside D, if there were one, we'd have $u_{xx} \leq 0$ and $u_{yy} \leq 0$. (This is the second derivative test of calculus.) So $u_{xx} + u_{yy} \leq 0$. At most maximum points, $u_{xx} < 0$ and $u_{yy} < 0$. So we'd get a contradiction to Laplace's equation. However, since it *is* possible that $u_{xx} = 0 = u_{yy}$ at a maximum point, we have to work a little harder to get a proof.

Here we go. Let $\epsilon > 0$. Let $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon|\mathbf{x}|^2$. Then, still in two dimensions, say,

$$\Delta v = \Delta u + \epsilon\Delta(x^2 + y^2) = 0 + 4\epsilon > 0 \quad \text{in } D.$$

But $\Delta v = v_{xx} + v_{yy} \leq 0$ at an interior maximum point, by the second derivative test in calculus! Therefore, $v(\mathbf{x})$ has no interior maximum in D.

Now $v(\mathbf{x})$, being a continuous function, has to have a maximum *somewhere* in the closure $\overline{D} = D \cup \text{bdy } D$. Say that the maximum of $v(\mathbf{x})$ is attained at $\mathbf{x}_0 \in \text{bdy } D$. Then, for all $\mathbf{x} \in D$,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{x}_0) = u(\mathbf{x}_0) + \epsilon|\mathbf{x}_0|^2 \leq \max_{\text{bdy } D} u + \epsilon l^2,$$

where l is the greatest distance from bdy D to the origin. Since this is true for any $\epsilon > 0$, we have

$$u(\mathbf{x}) \leq \max_{\text{bdy } D} u \qquad \text{for all } \mathbf{x} \in D. \tag{3}$$

Now this maximum is attained at some point $\mathbf{x}_M \in \text{bdy } D$. So $u(\mathbf{x}) \leq u(\mathbf{x}_M)$ for all $\mathbf{x} \in \overline{D}$, which is the desired conclusion.

The existence of a minimum point x_m is similarly demonstrated. (The absence of such points inside D will be proved by a different method in Section 6.3.) □

UNIQUENESS OF THE DIRICHLET PROBLEM

To prove the uniqueness, suppose that

$$\begin{aligned} \Delta u &= f \quad \text{in } D & \qquad \Delta v &= f \quad \text{in } D \\ u &= h \quad \text{on bdy } D & v &= h \quad \text{on bdy } D. \end{aligned}$$

We want to show that $u \equiv v$ in D. So we simply subtract equations and let $w = u - v$. Then $\Delta w = 0$ in D and $w = 0$ on bdy D. By the maximum principle

$$0 = w(\mathbf{x}_m) \le w(\mathbf{x}) \le w(\mathbf{x}_M) = 0 \quad \text{for all } \mathbf{x} \in D.$$

Therefore, both the maximum and minimum of $\boldsymbol{w}(\mathbf{x})$ are zero. This means that $w \equiv 0$ and $u \equiv v$.

INVARIANCE IN TWO DIMENSIONS

The Laplace equation is invariant under all rigid motions. A rigid motion in the plane consists of translations and rotations. A *translation* in the plane is a transformation

$$x' = x + a \qquad y' = y + b.$$

Invariance under translations means simply that $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$.

A *rotation* in the plane through the angle α is given by

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned} \tag{4}$$

By the chain rule we calculate

$$\begin{aligned} u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha \\ u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha \\ u_{xx} &= (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\cos\alpha - (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{y'}\sin\alpha \\ u_{yy} &= (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha. \end{aligned}$$

Adding, we have

$$\begin{aligned} u_{xx} + u_{yy} &= (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) + u_{x'y'} \cdot (0) \\ &= u_{x'x'} + u_{y'y'}. \end{aligned}$$

This proves the invariance of the Laplace operator. In engineering the laplacian Δ is a model for *isotropic* physical situations, in which there is no preferred direction.

The rotational invariance suggests that the two-dimensional laplacian

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

should take a particularly simple form in *polar coordinates*. The transformation

$$x = r\cos\theta \qquad y = r\sin\theta$$

has the jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

with the inverse matrix

$$\mathcal{J}^{-1} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix}.$$

(Beware, however, that $\partial r/\partial x \neq (\partial x/\partial r)^{-1}$.) So by the chain rule we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned}$$

These *operators* are squared to give

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]^2 \\ &= \cos^2\theta \frac{\partial^2}{\partial r^2} - 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \\ &\quad + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} \\ \frac{\partial^2}{\partial y^2} &= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)^2 \\ &= \sin^2\theta \frac{\partial^2}{\partial r^2} + 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} \\ &\quad + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}. \end{aligned}$$

(The last two terms come from differentiation of the coefficients.) Adding these operators, we get (lo and behold!)

$$\boxed{\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.} \tag{5}$$

It is also natural to look for special *harmonic functions that themselves are rotationally invariant*. In two dimensions this means that we use polar coordinates (r, θ) and look for solutions depending only on r. Thus by (5)

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r$$

if u does not depend on θ. This ordinary differential equation is easy to solve:

$$(ru_r)_r = 0, \quad ru_r = c_1, \quad u = c_1 \log r + c_2.$$

The function $\boldsymbol{\log r}$ will play a central role later.

INVARIANCE IN THREE DIMENSIONS

The three-dimensional laplacian is invariant under all rigid motions in space. To demonstrate its rotational invariance we repeat the preceding proof using vector-matrix notation. Any rotation in three dimensions is given by

$$\mathbf{x}' = B\mathbf{x},$$

where B is an *orthogonal* matrix (${}^tBB = B^tB = I$). The laplacian is $\Delta u = \Sigma_{i=1}^3 u_{ii} = \Sigma_{i,j=1}^3 \delta_{ij} u_{ij}$ where the subscripts on u denote partial derivatives. Therefore,

$$\begin{aligned}\Delta u &= \sum_{k,l} \left(\sum_{i,j} b_{ki}\delta_{ij}b_{lj} \right) u_{k'l'} = \sum_{k,l} \delta_{kl}\, u_{k'l'} \\ &= \sum_k u_{k'k'}\end{aligned}$$

because the new coefficient matrix is

$$\sum_{i,j} b_{ki}\delta_{ij}b_{lj} = \sum_i b_{ki}b_{li} = (B^tB)_{kl} = \delta_{kl}.$$

So in the primed coordinates Δu takes the usual form

$$\Delta u = u_{x'x'} + u_{y'y'} + u_{z'z'}.$$

For the three-dimensional laplacian

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

it is natural to use *spherical coordinates* (r, θ, ϕ) (see Figure 3). We'll use the notation

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2} \\ s &= \sqrt{x^2 + y^2} \\ x &= s\cos\phi \qquad z = r\cos\theta \\ y &= s\sin\phi \qquad s = r\sin\theta. \end{aligned}$$

(*Watch out:* In some calculus books the letters ϕ and θ are switched.) The calculation, which is a little tricky, is organized as follows. The chain of

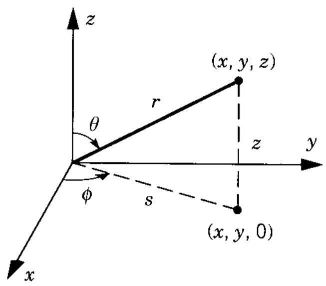

Figure 3

variables is $(x, y, z) \to (s, \phi, z) \to (r, \theta, \phi)$. By the two-dimensional Laplace calculation, we have both

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

and

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

We add these two equations, and cancel u_{ss}, to get

$$\begin{aligned}\Delta_3 &= u_{xx} + u_{yy} + u_{zz} \\ &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.\end{aligned}$$

In the last term we substitute $s^2 = r^2\sin^2\theta$ and in the next-to-last term

$$\begin{aligned}u_s &= \frac{\partial u}{\partial s} = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial \theta}{\partial s} + u_\phi\frac{\partial \phi}{\partial s} \\ &= u_r \cdot \frac{s}{r} + u_\theta \cdot \frac{\cos\theta}{r} + u_\phi \cdot 0.\end{aligned}$$

This leaves us with

$$\boxed{\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right],} \tag{6}$$

which may also be written as

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{7}$$

Finally, let's look for the special harmonic functions in three dimensions *which don't change under rotations*, that is, which depend only on r. By (7) they satisfy the ODE

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r} u_r.$$

So $(r^2 u_r)_r = 0$. It has the solutions $r^2 u_r = c_1$. That is, $u = -c_1 r^{-1} + c_2$. This important harmonic function

$$\frac{\mathbf{1}}{\mathbf{r}} = (x^2 + y^2 + z^2)^{-1/2}$$

is the analog of the special two-dimensional function $\log(x^2 + y^2)^{1/2}$ found before. Strictly speaking, neither function is finite at the origin. In electrostatics the function $u(\mathbf{x}) = r^{-1}$ turns out to be the electrostatic potential when a unit charge is placed at the origin. For further discussion, see Section 12.2.

EXERCISES

1. Show that a function which is a power series in the complex variable $x + iy$ must satisfy the Cauchy–Riemann equations and therefore Laplace's equation.
2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)
3. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u$, where k is a positive constant. (*Hint:* Look up Bessel's differential equation in [MF] or in Section 10.5.)
4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants. (*Hint:* Look for a solution depending only on r.)
5. Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.
6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.
7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.
8. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u = 0$ on $r = a$ and $\partial u/\partial r = 0$ on $r = b$. Then let $a \to 0$ in your answer and interpret the result.
9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at 100°C. Its outer boundary satisfies $\partial u/\partial r = -\gamma < 0$, where γ is a constant.
   (a) Find the temperature. (*Hint:* The temperature depends only on the radius.)
   (b) What are the hottest and coldest temperatures?
   (c) Can you choose γ so that the temperature on its outer boundary is 20°C?

10. Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D, $u = g$ on bdy D by the energy method. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.
11. Show that there is no solution of

$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$

in three dimensions, unless

$$\iiint_D f\,dx\,dy\,dz = \iint_{\text{bdy}(D)} g\,dS.$$

(*Hint:* Integrate the equation.) Also show the analogue in one and two dimensions.
12. Check the validity of the maximum principle for the harmonic function $(1 - x^2 - y^2)/(1 - 2x + x^2 + y^2)$ in the disk $\overline{D} = \{x^2 + y^2 \leq 1\}$. Explain.
13. A function $u(\mathbf{x})$ is *subharmonic* in D if $\Delta u \geq 0$ in D. Prove that its maximum value is attained on bdy D. [Note that this is not true for the minimum value.]

6.2 RECTANGLES AND CUBES

Special geometries can be solved by separating the variables. The general procedure is the same as in Chapter 4.

(i) Look for separated solutions of the PDE.
(ii) Put in the homogeneous boundary conditions to get the eigenvalues. This is the step that requires the special geometry.
(iii) Sum the series.
(iv) Put in the inhomogeneous initial or boundary conditions.

It is important to do it in this order: homogeneous BC first, inhomogeneous BC last.

We begin with

$$\Delta_2 u = u_{xx} + u_{yy} = 0 \quad \text{in } D, \tag{1}$$

where D is the rectangle $\{0 < x < a,\ 0 < y < b\}$ on each of whose sides one of the standard boundary conditions is prescribed (inhomogeneous Dirichlet, Neumann, or Robin).

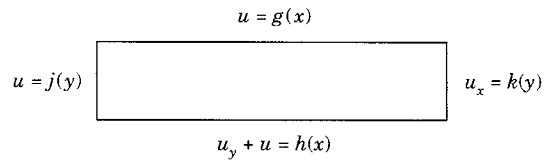

Figure 1

Example 1.

Solve (1) with the boundary conditions indicated in Figure 1. If we call the solution u with data (g, h, j, k), then $u = u_1 + u_2 + u_3 + u_4$ where u_1 has data $(g, 0, 0, 0)$, u_2 has data $(0, h, 0, 0)$, and so on. For simplicity, let's assume that $h = 0$, $j = 0$, and $k = 0$, so that we have Figure 2. Now we *separate variables* $u(x, y) = X(x) \cdot Y(y)$. We get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Hence there is a constant λ such that $X'' + \lambda X = 0$ for $0 \le x \le a$ and $Y'' - \lambda Y = 0$ for $0 \le y \le b$. Thus $X(x)$ satisfies a homogeneous one-dimensional problem which we well know how to solve: $X(0) = X'(a) = 0$. The solutions are

$$\boxed{\beta_n^2 = \lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} \quad (n = 0, 1, 2, 3, \ldots)} \tag{2}$$

$$X_n(x) = \sin \frac{(n + \frac{1}{2})\pi x}{a}. \tag{3}$$

Next we look at the y variable. We have

$$Y'' - \lambda Y = 0 \quad \text{with } Y'(0) + Y(0) = 0.$$

(We shall save the *inhomogeneous* BCs for the last step.) From the previous part, we know that $\lambda = \lambda_n > 0$ for some n. The Y equation has exponential solutions. As usual it is convenient to write them as

$$Y(y) = A \cosh \beta_n y + B \sinh \beta_n y.$$

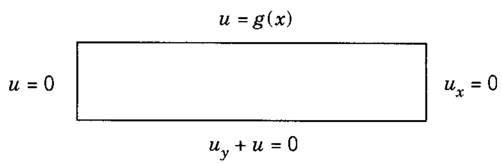

Figure 2

So $0 = Y'(0) + Y(0) = B\beta_n + A$. Without losing any information we may pick $B = -1$, so that $A = \beta_n$. Then

$$Y(y) = \beta_n \cosh \beta_n y - \sinh \beta_n y. \tag{4}$$

Because we're in the rectangle, this function is bounded. Therefore, the sum

$$\boxed{u(x, y) = \sum_{n=0}^{\infty} A_n \sin \beta_n x \; (\beta_n \cosh \beta_n y - \sinh \beta_n y)} \tag{5}$$

is a harmonic function in D that satisfies all three homogeneous BCs. The remaining BC is $u(x, \; b) = g(x)$. It requires that

$$g(x) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \cdot \sin \beta_n x$$

for $0 < x < a$. This is simply a Fourier series in the eigenfunctions $\sin \beta_n x$.

By Chapter 5, the coefficients are given by the formula

$$\boxed{A_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x \, dx.} \tag{6}$$

□

Example 2.

The same method works for a three-dimensional *box* $\{0 < x < a, 0 < y < b, 0 < z < c\}$ with boundary conditions on the six sides. Take Dirichlet conditions on a cube:

$$\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in } D$$

$$D = \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}$$

$$u(\pi, y, z) = g(y, z)$$

$$u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0.$$

To solve this problem we separate variables and use the five *homogeneous* boundary conditions:

$$u = X(x)Y(y)Z(z), \qquad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$X(0) = Y(0) = Z(0) = Y(\pi) = Z(\pi) = 0.$$

Each quotient X''/X, Y''/Y, and Z''/Z must be a constant. In the familiar way, we find

$$Y(y) = \sin my \quad (m = 1, 2, \ldots)$$

and

$$Z(z) = \sin nz \quad (n = 1, 2, \ldots),$$

so that

$$X'' = (m^2 + n^2)X, \qquad X(0) = 0.$$

Therefore,

$$X(x) = A\sinh(\sqrt{m^2 + n^2}\, x).$$

Summing up, our complete solution is

$$\boxed{u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}\, x)\sin my \sin nz.} \tag{7}$$

Finally, we plug in our inhomogeneous condition at $x = \pi$:

$$g(y, z) = \sum\sum A_{mn} \sinh(\sqrt{m^2 + n^2}\, \pi)\sin my \sin nz.$$

This is a *double* Fourier sine series in the variables y and z! Its theory is similar to that of the single series. In fact, the eigenfunctions $\{\sin my \cdot \sin nz\}$ are mutually orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$ (see Exercise 2). Their normalizing constants are

$$\int_0^{\pi}\int_0^{\pi} (\sin my \sin nz)^2\, dy\, dz = \frac{\pi^2}{4}.$$

Therefore,

$$A_{mn} = \frac{4}{\pi^2 \sinh(\sqrt{m^2 + n^2}\, \pi)} \int_0^{\pi}\int_0^{\pi} g(y, z)\sin my \sin nz\, dy\, dz. \tag{8}$$

Hence the solutions can be expressed as the doubly infinite series (7) with the coefficients A_{mn}. The complete solution to Example 2 is (7) and (8). With such a series, as with a double integral, one has to be careful about the order of summation, although in most cases any order will give the correct answer. □

EXERCISES

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:

$$\begin{aligned} u_x &= -a \quad \text{on } x = 0 \qquad & u_x &= 0 \quad \text{on } x = a \\ u_y &= b \quad \text{on } y = 0 \qquad & u_y &= 0 \quad \text{on } y = b. \end{aligned}$$

(*Hint:* Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.
3. Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \quad \text{for } y = 0 \text{ and for } y = \pi, \quad u = 0 \quad \text{for } x = 0 \quad \text{and}$$
$$u = \cos^2 y = \tfrac{1}{2}(1 + \cos 2y) \quad \text{for } x = \pi.$$

4. Find the harmonic function in the square $\{0 < x < 1, 0 < y < 1\}$ with the boundary conditions $u(x, 0) = x$, $u(x, 1) = 0$, $u_x(0, y) = 0$, $u_x(1, y) = y^2$.
5. Solve Example 1 in the case $b = 1$, $g(x) = h(x) = k(x) = 0$ but $j(x)$ an arbitrary function.
6. Solve the following Neumann problem in the cube $\{0 < x < 1, 0 < y < 1, 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.
7. (a) Find the harmonic function in the semi-infinite strip $\{0 \le x \le \pi, 0 \le y < \infty\}$ that satisfies the "boundary conditions":

$$u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = h(x), \quad \lim_{y\to\infty} u(x, y) = 0.$$

   (b) What would go awry if we omitted the condition at infinity?

6.3 POISSON'S FORMULA

A much more interesting case is the *Dirichlet problem for a circle*. The rotational invariance of Δ provides a hint that the circle is a natural shape for harmonic functions.

Let's consider the problem

$$\boxed{\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } x^2 + y^2 < a^2 \\ u &= h(\theta) \qquad \text{for } x^2 + y^2 = a^2 \end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

with radius a and any boundary data $h(\theta)$.

Our method, naturally, is to separate variables in *polar* coordinates: $u = R(r)\,\Theta(\theta)$ (see Figure 1). From (6.1.5) we can write

$$\begin{aligned} 0 = u_{xx} + u_{yy} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ &= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''. \end{aligned}$$

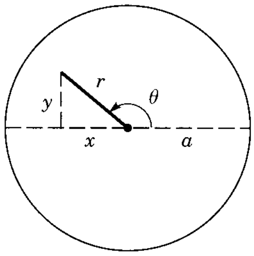

Figure 1

Dividing by $R\Theta$ and multiplying by r^2, we find that

$$\Theta'' + \lambda\Theta = 0 \tag{3}$$

$$r^2R'' + rR' - \lambda R = 0. \tag{4}$$

These are ordinary differential equations, easily solved. What boundary conditions do we associate with them?

For $\Theta(\theta)$ we naturally require periodic BCs:

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for } -\infty < \theta < \infty. \tag{5}$$

Thus

$$\lambda = n^2 \qquad \text{and} \qquad \Theta(\theta) = A\cos n\theta + B\sin n\theta \qquad (n = 1, 2, \ldots). \tag{6}$$

There is also the solution $\lambda = 0$ with $\Theta(\theta) = A$.

The equation for R is also easy to solve because it is of the Euler type with solutions of the form $R(r) = r^\alpha$. Since $\lambda = n^2$ it reduces to

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0 \tag{7}$$

whence $\alpha = \pm n$. Thus $R(r) = Cr^n + Dr^{-n}$ and we have the separated solutions

$$u = \left(Cr^n + \frac{D}{r^n}\right)(A\cos n\theta + B\sin n\theta) \tag{8}$$

for $n = 1, 2, 3, \ldots$. In case $n = 0$, we need a second linearly independent solution of (4) (besides $R =$ constant). It is $R = \log r$, as one learns in ODE courses. So we also have the solutions

$$u = C + D\log r. \tag{9}$$

(They are the same ones we observed back at the beginning of the chapter.)

All of the solutions (8) and (9) we have found are harmonic functions in the disk D, except that half of them are infinite at the origin ($r = 0$). But we haven't yet used any boundary condition at all in the r variable. The interval is $0 < r < a$. At $r = 0$ some of the solutions (r^{-n} and $\log r$) are infinite: We reject them. *The requirement that they are finite is the "boundary condition"* *at* $r = 0$. Summing the remaining solutions, we have

$$\boxed{u = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).} \tag{10}$$

Finally, we use the inhomogeneous BCs at $r = a$. Setting $r = a$ in the series above, we require that

$$h(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta).$$

This is precisely the full Fourier series for $h(\theta)$, so we know that

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi \tag{11}$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi. \tag{12}$$

Equations (10) to (12) constitute the full solution of our problem. □

Now comes an amazing fact. *The series* (10) *can be summed explicitly!* In fact, let's plug (11) and (12) directly into (10) to get

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} h(\phi) \frac{d\phi}{2\pi} \\ &\quad + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi)\{\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\}\, d\phi \\ &= \int_0^{2\pi} h(\phi) \left\{ 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right\} \frac{d\phi}{2\pi}. \end{aligned}$$

The term in braces is *exactly* the series we summed before in Section 5.5 by writing it as a geometric series of complex numbers; namely,

$$\begin{aligned} 1 &+ \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\ &= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\ &= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

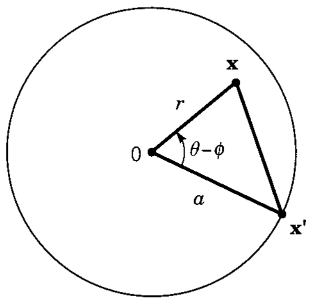

Figure 2

Therefore,

$$\boxed{u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,\frac{d\phi}{2\pi}.} \tag{13}$$

This single formula (13), known as *Poisson's formula*, replaces the triple of formulas (10)–(12). It expresses any harmonic function inside a circle in terms of its boundary values.

The Poisson formula can be written in a more geometric way as follows. Write $\mathbf{x} = (x, y)$ as a point with polar coordinates (r, θ) (see Figure 2). We could also think of $\mathbf{x}$ as the vector from the origin $\mathbf{0}$ to the point (x, y). Let $\mathbf{x}'$ be a point on the boundary.

$$\mathbf{x}\text{:}\quad \text{polar coordinates } (r,\theta)$$
$$\mathbf{x}'\text{:}\quad \text{polar coordinates } (a,\phi).$$

The origin and the points $\mathbf{x}$ and $\mathbf{x}'$ form a triangle with sides $r = |\mathbf{x}|$, $a = |\mathbf{x}'|$, and $|\mathbf{x} - \mathbf{x}'|$. By the law of cosines

$$|\mathbf{x} - \mathbf{x}'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi).$$

The arc length element on the circumference is $ds' = a\,d\phi$. Therefore, Poisson's formula takes the alternative form

$$\boxed{u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a}\int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2}\,ds'} \tag{14}$$

for $\mathbf{x} \in D$, where we write $u(\mathbf{x}') = h(\phi)$. This is a line integral with respect to arc length $ds' = a\,d\phi$, since $s' = a\phi$ for a circle. For instance, in electrostatics this formula (14) expresses the value of the electric potential due to a given distribution of charges on a cylinder that are uniform along the length of the cylinder.

A careful mathematical statement of Poisson's formula is as follows. Its proof is given below, just prior to the exercises.

Theorem 1. Let $h(\phi) = u(\mathbf{x}')$ be any continuous function on the circle $C = \text{bdy}\, D$. Then the Poisson formula (13), or (14), provides the only harmonic function in D for which

$$\lim_{\mathbf{x}\to\mathbf{x}_0} u(\mathbf{x}) = h(\mathbf{x}_0) \qquad \text{for all } \mathbf{x}_0 \in C. \tag{15}$$

This means that $u(\mathbf{x})$ is a continuous function on $\overline{D} = D \cup C$. It is also differentiable to all orders inside D.

The Poisson formula has several important consequences. The key one is the following.

MEAN VALUE PROPERTY

Let u be a harmonic function in a disk D, continuous in its closure $\overline{D}$. Then *the value of u at the center of D equals the average of u on its circumference*.

Proof. Choose coordinates with the origin $\mathbf{0}$ at the center of the circle. Put $\mathbf{x} = \mathbf{0}$ in Poisson's formula (14), or else put $r = 0$ in (13). Then

$$u(\mathbf{0}) = \frac{a^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{a^2}\, ds'.$$

This is the average of u on the circumference $|\mathbf{x}'| = a$.

MAXIMUM PRINCIPLE

This was stated and partly proved in Section 6.1. *Here is a complete proof of its strong form.* Let $u(\mathbf{x})$ be harmonic in D. The maximum is attained somewhere (by the continuity of u on $\overline{D}$), say at $\mathbf{x}_M \in \overline{D}$. We have to show that $\mathbf{x}_M \notin D$ unless $u \equiv$ constant. By definition of M, we know that

$$u(\mathbf{x}) \le u(\mathbf{x}_M) = M \qquad \text{for all } \mathbf{x} \in D.$$

We draw a circle around $\mathbf{x}_M$ entirely contained in D (see Figure 3). By the mean value property, $u(\mathbf{x}_M)$ is equal to its average around the circumference. Since the average is no greater than the maximum, we have the string of inequalities

$$M = u(\mathbf{x}_M) = \text{ average on circle } \le M.$$

Therefore, $u(\mathbf{x}) = M$ for all $\mathbf{x}$ on the circumference. This is true for any such circle. So $u(\mathbf{x}) = M$ for all $\mathbf{x}$ in the diagonally shaded region (see Figure 3). Now we repeat the argument with a different center. We can fill the whole domain up with circles. In this way, using the assumption that D is connected, we deduce that $u(\mathbf{x}) \equiv M$ throughout D. So $u \equiv$ constant.

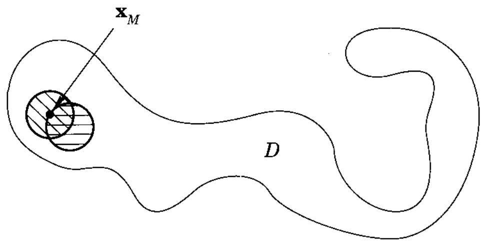

Figure 3

DIFFERENTIABILITY

Let u be a harmonic function in any open set D of the plane. Then $u(\mathbf{x}) = u(x, y)$ *possesses all partial derivatives of all orders in D.*

This means that $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$, $\partial^{100} u/\partial x^{100}$, and so on, exist automatically. Let's show this first for the case where D is a disk with its center at the origin. Look at Poisson's formula in its second form (14). The integrand is differentiable to all orders for $\mathbf{x} \in D$. Note that $\mathbf{x}' \in \text{bdy } D$ so that $\mathbf{x} \neq \mathbf{x}'$. By the theorem about differentiating integrals (Section A.3), we can differentiate under the integral sign. So $u(\mathbf{x})$ is differentiable to any order in D.

Second, let D be any domain at all, and let $\mathbf{x}_0 \in D$. Let B be a disk contained in D with center at $\mathbf{x}_0$. We just showed that $u(\mathbf{x})$ is differentiable inside B, and hence at $\mathbf{x}_0$. But $\mathbf{x}_0$ is an arbitrary point in D. So u is differentiable (to all orders) at all points of D.

This differentiability property is similar to the one we saw in Section 3.5 for the one-dimensional diffusion equation, but of course it is not at all true for the wave equation.

PROOF OF THE LIMIT (15)

We begin the proof by writing (13) in the form

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta - \phi)\, h(\phi) \frac{d\phi}{2\pi} \tag{16}$$

for $r < a$, where

$$P(r, \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n\theta \tag{17}$$

is the Poisson kernel. Note that P has the following three properties.

(i) $P(r, \theta) > 0$ for $r < a$. This property follows from the observation that $a^2 - 2ar\cos\theta + r^2 \geq a^2 - 2ar + r^2 = (a - r)^2 > 0$.

(ii)

$$\int_0^{2\pi} P(r,\theta)\frac{d\theta}{2\pi} = 1.$$

This property follows from the second part of (17) because $\int_0^{2\pi} \cos n\theta \, d\theta = 0$ for $n = 1, 2, \ldots$.

(iii) $P(r,\theta)$ is a harmonic function inside the circle. This property follows from the fact that each term $(r/a)^n \cos n\theta$ in the series is harmonic and therefore so is the sum.

Now we can differentiate under the integral sign (as in Appendix A.3) to get

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= \int_0^{2\pi} \left(P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta}\right)(r,\theta-\phi)\,h(\phi)\frac{d\phi}{2\pi} \\ &= \int_0^{2\pi} 0 \cdot h(\phi)\,d\phi = 0 \end{aligned}$$

for $r < a$. So u is harmonic in D.

So it remains to prove (15). To do that, fix an angle θ_0 and consider a radius r near a. Then we will estimate the difference

$$u(r,\theta_0) - h(\theta_0) = \int_0^{2\pi} P(r,\theta_0-\phi)[h(\phi) - h(\theta_0)]\frac{d\phi}{2\pi} \tag{18}$$

by Property (ii) of P. But $P(r,\theta)$ is concentrated near $\theta = 0$. This is true in the precise sense that, for $\delta \le \theta \le 2\pi - \delta$,

$$|P(r,\theta)| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a-r)^2 + 4ar\sin^2(\theta/2)} < \epsilon \tag{19}$$

for r sufficiently close to a. Precisely, for each (small) $\delta > 0$ and each (small) $\epsilon > 0$, (19) is true for r sufficiently close to a. Now from Property (i), (18), and (19), we have

$$|u(r,\theta_0) - h(\theta_0)| \le \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta_0-\phi)\,\epsilon\frac{d\phi}{2\pi} + \epsilon\int_{|\phi-\theta_0|>\delta} |h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi} \tag{20}$$

for r sufficiently close to a. The ϵ in the first integral came from the continuity of h. In fact, there is some $\delta > 0$ such that $|h(\phi) - h(\theta_0)| < \epsilon$ for $|\phi - \theta_0| < \delta$. Since the function $|h| \le H$ for some constant H, and in view of Property (ii), we deduce from (20) that

$$|u(r,\theta_0) - h(\theta_0)| \le (1 + 2H)\epsilon$$

provided r is sufficiently close to a. This is relation (15).

EXERCISES

1. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.
   (a) Find the maximum value of u in $\overline{D}$.
   (b) Calculate the value of u at the origin.
2. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition
$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$
3. Same for the boundary condition $u = \sin^3\theta$. (*Hint:* Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)
4. Show that $P(r, \theta)$ is a harmonic function in D by using polar coordinates. That is, use (6.1.5) on the first expression in (17).

6.4 CIRCLES, WEDGES, AND ANNULI

The technique of separating variables in polar coordinates works for domains whose boundaries are made up of concentric circles and rays. The purpose of this section is to present several examples of this type. In each case we get the expansion as an infinite series. (But summing the series to get a Poisson-type formula is more difficult and works only in special cases.) The geometries we treat here are

A wedge: $\{0 < \theta < \theta_0, 0 < r < a\}$
An annulus: $\{0 < a < r < b\}$
The exterior of a circle: $\{a < r < \infty\}$

We could do Dirichlet, Neumann, or Robin boundary conditions. This leaves us with a lot of possible examples!

Example 1. The Wedge

Let us take the wedge with three sides $\theta = 0$, $\theta = \beta$, and $r = a$ and solve the Laplace equation with the homogeneous Dirichlet condition on the straight sides and the inhomogeneous Neumann condition on the curved side (see Figure 1). That is, using the notation $u = u(r, \theta)$, the BCs are

$$\boxed{u(r, 0) = 0 = u(r, \beta), \qquad \frac{\partial u}{\partial r}(a, \theta) = h(\theta).} \tag{1}$$

The separation-of-variables technique works just as for the circle, namely,

$$\Theta'' + \lambda\Theta = 0, \qquad r^2R'' + rR' - \lambda R = 0.$$

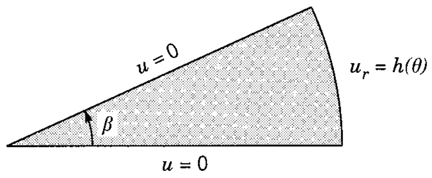

Figure 1

So the homogeneous conditions lead to

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = \Theta(\beta) = 0. \tag{2}$$

This is our standard eigenvalue problem, which has the solutions

$$\lambda = \left(\frac{n\pi}{\beta}\right)^2, \qquad \Theta(\theta) = \sin\frac{n\pi\theta}{\beta} \tag{3}$$

As in Section 6.3, the radial equation

$$r^2R'' + rR' - \lambda R = 0 \tag{4}$$

is an ODE with the solutions $R(r) = r^\alpha$, where $\alpha^2 - \lambda = 0$ or $\alpha = \pm\sqrt{\lambda} = \pm n\pi/\beta$. The negative exponent is rejected again because we are looking for a solution $u(r, \theta)$ that is continuous in the wedge as well as its boundary: the function $r^{-n\pi/\beta}$ is infinite at the origin (which is a boundary point of the wedge). Thus we end up with the series

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} A_n\, r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.} \tag{5}$$

Finally, the inhomogeneous boundary condition requires that

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{-1+n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

This is just a Fourier sine series in the interval $[0, \beta]$, so its coefficients are given by the formula

$$\boxed{A_n = a^{1-n\pi/\beta} \frac{2}{n\pi} \int_0^\beta h(\theta) \sin\frac{n\pi\theta}{\beta}\, d\theta.} \tag{6}$$

The complete solution is given by (5) and (6). □

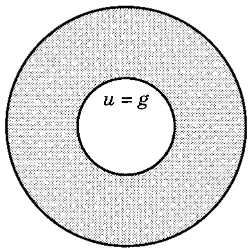

Figure 2

Example 2. The Annulus

The Dirichlet problem for an annulus (see Figure 2) is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u &= g(\theta) && \text{for } x^2 + y^2 = a^2 \\ u &= h(\theta) && \text{for } x^2 + y^2 = b^2 \end{aligned}$$

The separated solutions are just the same as for a circle except that we don't throw out the functions r^{-n} and $\log r$, as these functions are perfectly finite within the annulus. So the solution is

$$\boxed{\begin{aligned} u(r, \theta) = \tfrac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \cos n\theta \\ + (A_n r^n + B_n r^{-n}) \sin n\theta. \end{aligned}} \tag{7}$$

The coefficients are determined by setting $r = a$ and $r = b$ (see Exercise 3). □

Example 3. The Exterior of a Circle

The Dirichlet problem for the exterior of a circle (see Figure 3) is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{for } x^2 + y^2 > a^2 \\ u &= h(\theta) && \text{for } x^2 + y^2 = a^2 \end{aligned}$$

$$u \text{ bounded as } x^2 + y^2 \to \infty.$$

We follow the same reasoning as in the interior case. But now, instead of finiteness at the *origin*, we have imposed boundedness at *infinity*. Therefore, r^{+n} is excluded and r^{-n} is retained. So we have

$$u(r, \theta) = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta). \tag{8}$$

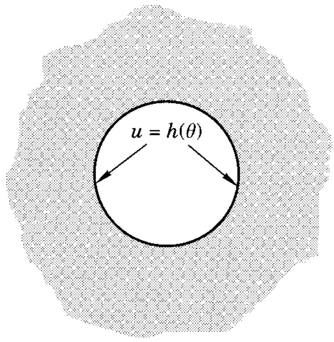

Figure 3

The boundary condition means

$$h(\theta) = \tfrac{1}{2}A_0 + \sum a^{-n}(A_n \cos n\theta + B_n \sin n\theta),$$

so that

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta \, d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta \, d\theta.$$

This is the complete solution but it is one of the rare cases when the series can actually be summed. Comparing it with the interior case, we see that the only difference between the two sets of formulas is that r and a are replaced by r^{-1} and a^{-1}. Therefore, we get Poisson's formula with only this alteration. The result can be written as

$$\boxed{u(r,\theta) = (r^2 - a^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\,\frac{d\phi}{2\pi}} \tag{9}$$

for $r > a$. □

These three examples illustrate the technique of separating variables in polar coordinates. A number of other examples are given in the exercises. What is the most general domain that can be treated by this method?

EXERCISES

1. Solve $u_{xx} + u_{yy} = 0$ in the *exterior* $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.

2. Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition
$$\frac{\partial u}{\partial r} - hu = f(\theta),$$
where $f(\theta)$ is an arbitrary function. Write the answer in terms of the Fourier coefficients of $f(\theta)$.
3. Determine the coefficients in the annulus problem of the text.
4. Derive Poisson's formula (9) for the exterior of a circle.
5. (a) Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, whose outer edge ($r = 2$) is insulated, and on whose inner edge ($r = 1$) the temperature is maintained as $\sin^2 \theta$. (Find explicitly all the coefficients, etc.)
   (b) Same, except $u = 0$ on the outer edge.
6. Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter ($\theta = 0, \pi$) and
$$u = \pi \sin \theta - \sin 2\theta \quad \text{on } r = 1.$$
7. Solve the problem $u_{xx} + u_{yy} = 0$ in D, with $u = 0$ on the two straight sides, and $u = h(\theta)$ on the arc, where D is the wedge of Figure 1, that is, a sector of angle β cut out of a disk of radius a. Write the solution as a series, but don't attempt to sum it.
8. An annular plate with inner radius a and outer radius b is held at temperature B at its outer boundary and satisfies the boundary condition $\partial u/\partial r = A$ at its inner boundary, where A and B are constants. Find the temperature if it is at a steady state. (*Hint:* It satisfies the two-dimensional Laplace equation and depends only on r.)
9. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a, 0 < \theta < \beta$ with the BCs
$$u = \theta \quad \text{on } r = a, \quad u = 0 \quad \text{on } \theta = 0, \quad \text{and} \quad u = \beta \quad \text{on } \theta = \beta.$$
(*Hint:* Look for a function independent of r.)
10. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:
$$u = 0 \quad \text{on } x = 0 \text{ and on } y = 0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 1 \quad \text{on } r = a.$$
Write the answer as an infinite series and write the first two nonzero terms explicitly.
11. Prove the uniqueness of the Robin problem
$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D,$$
where D is any domain in three dimensions and where a is a positive constant.

12. (a) Prove the following still stronger form of the maximum principle, called the Hopf form of the maximum principle. If $u(\mathbf{x})$ is a nonconstant harmonic function in a connected plane domain D with a smooth boundary that has a maximum at $\mathbf{x}_0$ (necessarily on the boundary by the strong maximum principle), then $\partial u/\partial n > 0$ at $\mathbf{x}_0$ where $\mathbf{n}$ is the unit *outward* normal vector. (This is difficult: see [PW] or [Ev].)
    (b) Use part (a) to deduce the uniqueness of the Neumann problem in a connected domain, up to constants.
13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.
14. Answer the last question in the text.