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What is Calculus About?

W. W. Sawyer

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WHAT IS CALCULUS ABOUT?

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WHAT IS CALCULUS ABOUT?

by

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2

MATHEMATICAL ASSOCIATION OF AMERICA

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CONTENTS

Guide to Further Study

As was emphasized at the beginning of this book, the ideas of calculus have great powers of growth and development; from this small root there come many branches of pure mathematics and of physical science. This growth will seem natural and orderly to anyone who traces its development from the root upwards. But the latest fruits of this growth may seem very strange and unnatural to someone who meets them suddenly without any knowledge of the tree from which they came. It is therefore extremly important to read books about calculus in *the correct order*. A student with real genius for mathematics might be reduced to despair, if he were required to read a modern text on analysis without any previous preparation. It would be a book in a foreign language; the words would not convey any ideas. This does not mean that the ideas, built up gradually and in the right order, are particularly difficult.

One can recognize three stages in the development of calculus, which fit rather neatly into the changes of century.

(1) 1600–1800. The happy-go-lucky stage. The main emphasis is on formulas and results.
(2) 1800–1900. The analysis or epsilon-delta stage.
(3) 1900– . The stage of abstraction and extreme generalization.

In passing from each stage to the next, new ideas and a new way of thinking have to be learned. A student may experience some kind of crisis. At first, he feels he cannot grasp these new ideas. If he keeps reading about them in different books, and thinks and works problems for himself, he should reach a stage where everything seems to fall into place; he then finds it hard to see why these ideas ever seemed difficult to him. He sees that the new ideas are simply the old ideas expressed in a different way, perhaps a little more clearly.

Before 1900, there was a general belief that calculus was much too difficult to teach young mathematicians. Many results that could have been proved very simply by calculus had to be obtained by more painful algebra. This procedure was known as "calculus dodging." Round about 1900, John Perry and others in England began to advocate the view that the essential ideas and methods of calculus were simple and could be taught in schools. E. H. Moore, professor of mathematics at Chicago and the father of modern American mathematics, gave this view his blessing.† One of Moore's students, F. L. Griffin, pioneered the teaching of calculus to first-year college students. This was regarded as a very daring thing to do. Griffin's celebrated book, *Introduction to Mathematical Analysis,* shows how he went about this. The word "calculus" did not appear in the title, in case the students were terrified. This book, which deals both with trigonometry and calculus, and emphasizes the relation of calculus to physics and engineering, can be warmly recommended as an introduction to calculus.

In England, during the past fifty years, the teaching of calculus in high schools has become the general practice. It is not possible to say just how many years of calculus are done in school as, in the better schools, students are encouraged to work ahead at their own pace. Whether a student meets calculus at 18 or 16 or 14 depends largely on his own ability. As there are no books written in America for teaching calculus to 15-year-olds, it may be of interest to mention some English calculus texts. Being much smaller and more concise and plainer than American texts, these books are very inexpensive.

Introductory calculus ideas are usually brought in towards the end of the algebra text. See, for example, Durell, Palmer, and Wright, *Elementary Algebra* (Bell, Portugal Street, London, W.C.2).

Fawdry and Durell, *Calculus for Schools* (Arnold, London) gives a very simple introduction to calculus.

Durell and Robson, *Elementary Calculus,* volumes I and II (Bell), introduces calculus in a simple way; the authors make great efforts not to make any statement that the student will find to be untrue when he reaches a more advanced stage. This book takes the student further into calculus than *Calculus for Schools* does. Volume II explains, among other things, the idea of *partial differentiation,* which is of importance for further mathematics and, in particular, for mathematical physics.

† See the first yearbook of the National Council of Teachers of Mathematics (U.S.A.).

However, you will need to work far more exercises than are given in this book, if you are to remember what you read. You should seek out exercises from any book on calculus† and work at them in such a way that you always keep in good practice.

Piaggio, *Differential Equations* (Bell) is a very readable book that might follow *Elementary Calculus*, Part II. The earlier chapters, in particular, do not delve into underlying theory but give the student an idea of how calculus is used. Chapter IV gives a very quick and simple introduction to Fourier series.

We now wish to pass from the happy-go-lucky stage to the epsilon-delta stage. Most texts do this far too suddenly. The book that takes the student most gradually and carefully from the old to the new viewpoint seems to me to be Hardy, *Pure Mathematics* (Cambridge University Press).

You may wonder why we call this the "epsilon-delta" stage. In the 19th century, many ideas which had been previously accepted as sufficiently clear—for example, "continuous", "approaches"—were carefully analyzed and defined. The new definitions usually contained the phrase "given any positive ϵ, however small, δ can be found such that ...".‡ People came to think of this phrase as typical of the new analysis.

It may help you to acquire these new ideas if you read some general accounts of how mathematics developed in this direction; for example:

Tobias Dantzig, *Number, the Language of Science* (Doubleday Anchor, 95 cents). Especially, Chapters 7, 8, 9.

Felix Klein, *Elementary Mathematics from an Advanced Viewpoint; Arithmetic, Algebra, Analysis* (Dover).

W. W. Sawyer, *Mathematician's Delight* (Penguin, 85 cents).

Once you have reached the stage where you can read a book written in the epsilon-delta language, there is no doubt what your next book should be—Courant, *Differential and Integral Calculus* (Interscience, N.Y.). This book is admirably clear. As Nathan G. Park says, in his *Guide to the Literature of Mathematics and Physics*, "Courant will give the student the best possible balance between vigor and rigor."

Where you go after this must depend very much on your personal tastes and aims. There is so much mathematics that, unfortunately, no one can learn the whole of it.

† A bad book can contain good exercises. For example, J. Edwards, *The Differential Calculus* (St. Martin's Press), is famous for the number of illogical and untrue statements it contains. But it contains an amazing collection of examples for anyone wishing to master formal manipulation in calculus.

‡ The symbols δ, ϵ are read "delta" and "epsilon". They are the letters of the Greek alphabet corresponding to our *d* and *e*.

In every branch of mathematics there are a few central ideas; in the following-out of these central ideas, all kinds of detailed investigations become necessary. Very many books give you the details without the central ideas that illuminate the whole subject. You should not therefore be disturbed if, when trying to learn a new branch of mathematics, you find the books on it completely incomprehensible. Continue to search in libraries or bookshops until you find a book that gives you the essential ideas. Sometimes you cannot find one book that gives you all you want; you may have to pick up a clue here and a clue there.

Some of the mathematics of the 20th century, in particular, seems most strange if you are suddenly plunged into the middle of it. It appears to be an entirely different subject from the mathematics you learned at school. Yet it grew from the older mathematics. This happened in something like the following way. The older mathematics dealt in the main with definite objects. You had to solve a particular equation, or prove a theorem about some particular shape in geometry, or study the vibrations of a particular mechanical system. As time passed, more and more special results about particular objects accumulated, and mathematicians began to long for some way of systematizing the subject. There were too many details for anyone to remember them all. Then it began to be noticed that, very often, the details were only obscuring the picture. Of all the information available about some object, only a small part might be necessary for solving the problem in hand; that aspect was helpful, all the rest was merely distracting. Mathematicians began to study these special aspects, much as a chemist might extract a vitamin from a complex substance. Someone who knew nothing about vitamin pills might not realize that such a thing was food at all. In the same way, a person new to modern abstract mathematics might not realize that it was mathematics at all.

This extraction of the essential ideas was also made necessary by mathematicians going on to more and more complicated problems. Some vibrations in mechanics can be represented by the motion of a point in two or three dimensions. We are able to visualize the mechanical problem by means of geometry. Some more complicated problems require four or five or six or more dimensions to visualize. So we develop the geometry of n dimensions, and this helps us to visualize the problem, in a somewhat vaguer way, by the analogy with ordinary space of 3 dimensions. Some problems require an infinity of dimensions. Now space of infinite dimensions in some ways resembles space of three dimensions, and in some ways differs from it. So it becomes necessary to separate very carefully those ideas we have about ordinary geometry which are still true and helpful when we are thinking about infinite dimensional space, from those which are untrue and misleading when used as analogies. By this kind of road mathematicians reached the concept of *Hilbert space.*

The connection between physics and the geometry of space is brought out very nicely in Courant and Hilbert, *Methods of Mathematical Physics* (Interscience, N.Y.), Volume I.

A book which carries the reader from 19th- to 20th-century mathematics, without any sense of a sudden break, is Riesz and Nagy, *Functional Analysis* (Ungar, N.Y., 1955).

By contrast, one may mention Munroe, *Introduction to Measure and Integration.* This book from the start has the flavor of the 20th century. For a reader with the necessary background it is extremely clear.

E. J. McShane, *Integration* (Princeton) is written for "students of little maturity" who are beginning graduate work in mathematics. Any student of strong mathematical ability will, of course, be able to read it several years earlier than this.

A student who finds difficulty in passing from traditional calculus to the set-theoretical approach may find something of interest in the huge book, Hobson, *Functions of a Real Variable* (Cambridge University Press, reprint by Dover). This book has been described as a strange mixture of careful rigor and astonishing errors. It was written in the years when the new theories were coming in, so you see Hobson (who had grown up under the older approach) trying to explain to himself and others what these new ideas are. It is a book to browse in, rather than to read from cover to cover. The fact that the book contains errors† is valuable. It means that you cannot accept any statement on authority; all the time, you have to ask yourself, "Do I believe this?"

† See Littlewood, *A Mathematician's Miscellany* (Methuen, London), page 68.

List of Technical Terms

Throughout this book, I have explained things as far as possible in everyday language. When you read other books on calculus, you will need to know the symbols and the special names that mathematicians use.

Derivative. s' is called the derivative of s. You may also find the symbols ds/dt, Ds, $D_t s$ used for the derivative. These have exactly the same meaning as s'.

Differentiation. The problem of finding the derivative is called *differentiation.* Thus in Chapter 3 you learned how to differentiate t^2, in Chapter 4 how to differentiate t^n, and in Chapter 5 how to differentiate any polynomial.

Integration. Finding an area or a volume is a problem of *integration.* Integration can be regarded as the reverse of differentiation. The symbol $\int$ is used in connection with integration. At the end of Chapter 9 we found the volume of a half sphere. A mathematician would write our result

$$\frac{2}{3}\pi = \int_0^1 \pi(1 - t^2)dt.$$

Limit. Many times in this book we have noticed that something "approached" or "seemed to be settling down to" a certain value. In Chapter 2, the numbers 5, 5.9, 5.99, 5.999, ... seemed to be approaching the value 6. In Fig. 23 on page 54, the slope of the line

CD approached closer and closer to the slope of the curve at C. By writing enough ones, you can make $0.11111 \cdots 111$ approach as close as you like to the fraction 1/9. In each of these cases, something is tending *towards a limit*. While the word has not been stressed, the idea of limit runs through everything discussed in this book.

Function. The meaning of the word *function* has developed and changed during the last three centuries. At first, "y is a function of x" meant something very much like "y is related to x by some formula." This would cover, for example, $y = 2x + 1$ or $y = x^2$ or $y = \sqrt{x^3 + 1}$. In each of these cases, an 18th-century mathematician sees a formula giving y in terms of x. Suppose he has some procedure that can be applied to each of these, and to many other formulas as well. He does not mind what the particular formula is; he wants to lump them all together. He would say, "Let y be any function of x." He would write this, for short, as $y = f(x)$.

As time passed, this viewpoint proved insufficient. In Fig. 65, we had a graph consisting of parts of two lines. Between $x = 0$ and $x = 1$, the value of y was $1 - x$. For x larger than 1, the value of y was zero. So two formulas were involved, $y = 1 - x$ and $y = 0$. What shall we say? Do we have two functions here, or part of one function grafted onto part of another function, or what? There were furious discussions between mathematicians about this question. As time passed, more and more strange graphs came to the attention of mathematicians, and it was eventually decided that the best thing to do was to forget all about the simple formulas of algebra. Instead, it was decided to write $y = f(x)$ if any procedure whatever fixed the value of y so soon as the value of x was given. Thus the graph of Fig. 65 defines a function; if I tell you any positive number for x, you can read the corresponding value of y from the graph. If I say $x = 2$, you answer $y = 0$. If I say $x = \frac{3}{4}$, you answer $y = \frac{1}{4}$. You are never at a loss for an answer. As soon as I say the value of x, that fixes the value of y. Good; we do not inquire any further into the matter. Any procedure that associates a single value of y with each value of x defines a function.

The graph of Fig. 65 was drawn only for positive values of x. So the function is not defined for all values of x, but only for positive values. Mathematicians have decided that this is nothing to worry about. In first-year algebra, $y = \sqrt{x}$ is defined only for positive x. We do not know anything about the square root of a negative number. We accept this situation. We say that $\sqrt{x}$ is defined (in beginning algebra) only for the *domain* of positive values of x. If $y = f(x)$ is defined only for a certain set of values of x, these values are said to form the *domain of the function*.

For example, if x is a whole number, we can define y as the largest prime factor of x, but this definition would not make sense if x was a fraction. We have defined a function for the *domain* of the whole numbers.

In traditional algebra, x and y stand for numbers. But functions can be defined which have nothing to do with numbers. For example, suppose we consider all the instants of time since the year 1789. These form the domain of the function. At any instant during those years, the President of the United States had eyes of some color. With some historical research, one could find out what that color was, corresponding to each time. So we have a procedure for associating a definite color with each instant of time since 1789. This procedure defines a function. We have come a long way from algebraic formulas! The word "function" is generally used today in this very wide sense.

One point arises. Returning to ordinary algebra, we might consider the following two procedures.

Procedure I: Take any number, x.
Add 1 to it.
Square the result.
This gives the value of y.

Procedure II: Take any number, x.
Square it.
Add twice the number.
Add 1.
This gives the value of y.

Each procedure defines a function. The procedures are different. Shall we say that the resulting functions are different?

If we take for example $x = 5$, Procedure I gives $y = (5 + 1)^2 = 36$ and Procedure II gives $y = 5^2 + 2 \cdot 5 + 1 = 36$. So either procedure leads us to associate $y = 36$ with $x = 5$. And the same happens, of course, with any number you may choose. Procedure I corresponds to the formula $y = (x + 1)^2$, and Procedure II to the equivalent formula $y = x^2 + 2x + 1$.

Mathematicians have agreed to say that both procedures define the *same* function. We are only interested in the final result, not in the details of the calculation. If we have any procedure, which leads you to say $y = 36$ when I say $x = 5$, and makes you say $y = 4$ when I say $x = 1$, and quite generally makes you say $y = (n + 1)^2$ when I say $x = n$, then that procedure defines the same function as Procedure I above.

You may meet a definition of function which begins: "A function is a set of ordered pairs...". This is a very condensed and abstract way of saying what I have outlined above. I am not too happy myself with any definition that begins "A function is ...", any more than I should be happy with a definition that began "Electricity is...", or "Magnetism is...", or "Gold is...". I can give you a series of tests, in each case, that will enable you to say, "This is probably an electrically charged object", or "This is probably a magnet", or "This is probably a piece of gold". In the same way, I have given tests above that will enable you to tell (i) whether a particular procedure defines a function, and (ii) whether two apparently different procedures define the same function.

It is important to distinguish between the *function* and the *value of the function*. If f stands for the function defined by Procedure I above, we may write $36 = f(5)$. This means that 36 is the value of y that Procedure I leads us to associate with $x = 5$. 36 is called the *value of the function* for $x = 5$. It would be wrong to say that 36 *is the function* f. It would be nearer the truth to say that the letter f, by itself, indicates the operation of "adding 1 and then squaring". $f(5)$ represents the result when this operation is applied to the particular number 5.

Answers to Questions and Exercises

p. 12 1. The steeper the line is, the faster the object is moving.

p. 13 2. (a) is (ii) (d) is (iv)
(b) is (iii) (e) is (i)
(c) is (v)

3. (f) is (viii) (h) is (ix)
(g) is (vi) (i) is (vii).

p. 20 1. $s = 20t$. $s' = 20$.

2.

t	0	1	2	3
s	0	30	60	90

Velocity is 30 mph. $s' = 30$.

3. $s' = 40$.

4. 50.

5. k.

p. 20 (1) 10 (2) 10 (3) 10
If $s = 10t + c$, $s' = 10$.

p. 21 (4) 20 (5) 20 (6) 20 (7) 20
Conclusion: If $s = 20t + c$, $s' = 20$.

(8) 30 (9) 50 (10) 40 (11) 30 (12) 50
The illustrations are straight lines. If the scale is kept fixed, the larger the velocity s' is, the steeper the line will be. If different laws give the same velocity, as in (1), (2), (3), the lines will be parallel.

p. 33 1.

t	1	1.001	2	2.001	3	3.001	4	4.001	5	5.001
s	1	1.003	8	8.012	27	27.027	64	64.048	125	125.075

t	1	2	3	4	5
s'	3	12	27	48	75

Law: $v = s' = 3t^2$.

2. See p. 34.
3. See p. 34.
4. $5t^4$; $6t^5$; nt^{n-1}. See pp. 34–35.

p. 44 (1) $20t$ (4) $800t^{99}$
(2) $60t^2$ (5) $6t^2$
(3) $16t^3$ (6) $6t$

p. 45 (1) $20t + 60t^2$ (4) $35t^6 - 8t^3$
(2) $6t^2 + 6t$ (5) $20t + 60t^2 - 20t^3$
(3) $35t^6 + 8t^3$

p. 47 1. 0; 2; 2 5. $10t - 4$
2. 0; $3t^2$; $3t^2$ 6. $6t^2 - 6t - 10$
3. 0; 3; $2t$; $2t + 3$ 7. $80t^{19} + 30t^{14} - 30t^9 + 5$
4. $10t + 4$ 8. $60t^5 + 60t^4 - 60t^3 + 60t^3 - 60t + 60$

p. 52 (1) 1 (2) 1 (3) 2 (4) 3
p. 53 (5) -1 (6) -2

p. 68 6. $3x^2 - 6x + 9 = 3(x - 1)^2 + 6$, never zero and never negative. If $y = x^3 - 3x^2 + 9x$, $y' = 3x^2 - 6x + 9$. So y' is always positive. Curve uphill; resembles Fig. 37.

What is Calculus About?

Calculus, invented by Newton and Leibniz in the seventeenth century, has played a decisive role in the development of mathematics and the growth of our present technological society. It is an indispensable tool of both the pure and applied sciences and is one of the cornerstones of modern mathematics. It has provided ways of understanding such phenomena as the velocity of a moving object at any moment, the rate at which moving objects change their speeds, and, more generally, the way in which quantities vary as factors affecting them change. In this book, the author tells what calculus is about in simple nontechnical language, understandable to any interested reader.

W. W. Sawyer (April 5, 1911 – February 15, 2008) was born in St. Ives, Hunts, England. He attended Highgate School and St. John's College, Cambridge, where he specialized in the mathematics of quantum theory and relativity. Sawyer lectured at universities in Dundee, Manchester, Ghana, New Zealand, Illinois, Connecticut, and Toronto. He retired in 1976 from the University of Toronto. His books include *Mathematician's Delight*, *Prelude to Mathematics*, *A Concrete Approach to Abstract Algebra*, and *An Engineering Approach to Linear Algebra.*

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