WHEN UNDERSTANDING EVOKES APPRECIATION: THE EFFECT OF MATHEMATICS CONTENT KNOWLEDGE ON AESTHETIC PREDISPOSITION

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This study explored the problem solving experience of pre-service teachers in finding the greatest common factor and the least common multiple using many different approaches. In particular, it examined the effect of pre-service teachers' mathematics content knowledge on how they chose their preferred approach and how they valued the most efficient approach. The findings indicated that the most efficient approach was appreciated only if such approach was reasonably understood by these pre-service teachers.

INTRODUCTION

Aesthetic values play a central role in experts' mathematics problem solving experience (Silver & Metzger, 1989). Typically, a problem solving approach is considered "beautiful" if it is particularly clear, simple, and unexpected. Beginning problem solvers have also demonstrated the ability to develop and favour certain problem solving approaches often considered more efficient than others (Silver, Leung, & Cai, 1995). Nonetheless, little is known about the extent to which beginning problem solvers' mathematics content knowledge influenced how they chose their preferred approach and how they valued the most efficient approach. In particular, does an aesthetic appreciation for the most efficient approach necessitate certain understanding of that approach? Is it possible to appreciate the most efficient approach if one lacks the understanding of problem solving using many different approaches? Does knowing more than one approach allow for more flexibility in problem solving?

The purpose of this article is to investigate the effect of pre-service teachers' mathematics content knowledge on their aesthetic predisposition in their problem solving experience involving problems of finding the greatest common factor (GCF) and the least common multiple (LCM) of two numbers. It begins with the theoretical background on the benefits of problem solving using many different approaches and the mathematics aesthetic aspect of experts' problem solving practices, as well as examples of beginning problem solvers' conceptions of what it means for an approach to be efficient. In connection with the instruments used in the methodology, several approaches for finding the GCF and LCM are discussed. The article continues with the findings and consequent analysis, and concludes with pedagogical recommendations that promote the habit of mind of creative problem solving and the mathematics aesthetic appreciations.

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THEORETICAL BACKGROUND

In recent years, mathematics problems solving using many different approaches has drawn more attention than before (Leikin & Levav-Waynberg, 2007). Some researchers, in fact, considered such practice to be beneficial for students' mathematics learning experience.

Silver et al. (2005) believed that students "can learn more from solving one problem in many different ways than [they] can from solving many different problems, each in only one way" (p. 288). They particularly advised students interested in mathematics to obtain more experience in problem solving with many different approaches. They regarded such experience as having "the potential advantage of providing students with access to a range of representations and solution strategies in a particular instance that can be useful in future problem-solving encounters" (p. 288). They also considered the use of many different approaches in order to "facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 288).

Tabachneck, Koedinger, and Nathan (1994) recognized the purpose of adopting many different approaches in problem solving. They argued that on its own, each approach might entail disadvantages and weaknesses. In order to overcome these, they recommended students operate a combination of approaches, instead of counting on only one approach. More specifically, they maintained that students could benefit from employing this learning style in mathematical problem solving. In addition to teaching to solve one problem with many approaches, psychologists encouraged teaching a coherent interrelation among those approaches (Skemp, 1987; De Jong et al., 1998; Van Someren et al., 1998; Bodemer et al., 2004). Equally important, Reeves and Weisberg (1994) recommended showing students many analogical problems or examples concurrently. On the whole, cognitive psychologists took a positive stance on problem solving using many approaches, as did mathematics education researchers.

Given the many possible different approaches to solve the same problem, a decision to choose one approach over the many other approaches may be less than arbitrary. Aesthetic aspects were particularly considered in many studies connected with experts' preference in problem solving approaches.

Silver and Metzger (1989) assessed the role of the aesthetics in a study involving university professors in mathematics. They found that these expert problem solvers displayed signs of aesthetic emotion. On one occasion, a subject resisted the temptation to resort to the use of calculus in solving a geometry problem, acknowledging the possibility of a "messy equation" (p. 66). Only after some unsuccessful attempts to seek a geometric approach did the subject concede to solving the problem using calculus. Although successful, he felt that "calculus failed to satisfy his personal goal of understanding, as well as his aesthetic desire for 'harmony' between the elements of the problem and elegance of solution" (p. 66). On another occasion, having solved another geometry problem algebraically, the same subject appeared unsettled, recognizing that a geometric approach could be "more elegant" (p. 66).

Dreyfus and Eisenberg (1986) were interested in exploring whether students considered aesthetic values of mathematical reasoning in their problem solving approaches. Their study involved college-level mathematics students who had been rigorously prepared in advanced mathematics courses. They were tested on several carefully chosen mathematics problems which involved many different approaches not immediately apparent to average students, yet readily accessible with high school mathematics knowledge. After completing the test, students were presented with approaches that were considered elegant by expert mathematicians.

Dreyfus and Eisenberg (1986) discovered that not only were the students not able to supply elegant approaches in the test as they had been expected to, but they were also not able to recognize the differences between elegant and pedestrian approaches. Furthermore, when presented with elegant approaches, they showed no enthusiasm and found them no more attractive than their own approaches. In other words, they had no sense of aesthetic appreciation. Dreyfus and Eisenberg (1986) concluded that mathematics instruction in classroom settings lacked an emphasis on reflective thinking, especially aesthetic value.

Sinclair (2004) analysed the role of aesthetic values from several conceptual insights. She drew examples from existing empirical findings such as those by Dreyfus and Eisenberg (1986) and Silver and Metzger (1989). In one of her interpretations of their work, she suggested that "mathematicians' aesthetic choices might be at least partially learned from their community as they interact with other mathematicians and seek their approval" (Sinclair, 2004, p. 276). Furthermore, she indicated that mathematical beauty was only feasible in the process "when young mathematicians are having to join the community of professional mathematicians—and when aesthetic considerations are recognized (unlike at high school and undergraduate levels)" (p. 276).

Nevertheless, few studies have demonstrated that beginning problem solvers might actually be capable of recognizing mathematical "beauty" from the standpoint of efficiency. Nesher, Hershkovitz, and Novotna (2003) investigated students' choices of approaches to solving algebra problems. Specifically, they were interested in ninth grade students' use of independent variables when solving algebra word problems. These word problems involved a situation with three unknown quantities whose sum was known. In interviewing the students, the researchers found that the students' choices of independent variables were mainly influenced by the order in which the quantities were described in the word problems. At the same time, students favoured independent variables with the smallest quantity in relation to the other two quantities discussed in the problems. By doing so, students unconsciously revealed their natural inclination to working with whole numbers as opposed to rational numbers. To some extent, students were capable on their own of constructing the notions of the more efficient approach in problem solving.

METHODOLOGY

This study involved 37 pre-service teachers (31 female, 6 male, age 20-24) in an elementary (age 5-12, grade K-6) education program at a large, urban university. These 37 pre-service teachers were enrolled in a mathematics content course in which the researcher was the instructor. Four approaches for the GCF and four approaches for the LCM were introduced to these pre-service teachers during the instruction time of two 50-minutes sessions.

Using an example of finding the GCF and LCM of 24 and 36, the eight approaches are discussed as follows. The first approach for finding the GCF is the Set Intersection Method where given all factors of 24 (e.g., 1, 2, 3, 4, 6, 8, 12, and 24) and 36 (e.g., 1, 2, 3, 4, 6, 9, 12, 18, 36), the common factors of 24 and 36 are 1, 2, 3, 4, 6, and 12, of which 12 is the largest. The second approach for finding the GCF is the Prime Factorization Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the GCF consists of the prime factors with the smaller exponents (e.g., $2^2 \cdot 3^1 = 12$). The third approach for finding the GCF is the Repeated Subtractions Method where the GCF is obtained by repeatedly subtracting the smaller number from the larger number until both numbers are equal (e.g., GCF(24,36)=GCF(36-24,24)=GCF(12,24)=GCF(24-12,12)=GCF(12,12)=12). The fourth method for finding the GCF is the Euclidean Algorithm Method where the GCF is obtained by repeatedly dividing the larger number by the smaller number until a remainder of zero is obtained (e.g., 36÷24=1R12, 24÷12=2R0, and thus, GCF is 12).

The first approach for finding the LCM is the Set Intersection Method where given some multiples of 24 (e.g., 24, 48, 72, 96, 120, 144, …) and 36 (e.g., 36, 72, 108, 144, …), the common multiples of 24 and 36 are 72, 144, …, of which 72 is the smallest. The second approach for finding the LCM is the Prime Factorization Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the LCM consists of the prime factors with the larger exponents (e.g., $2^3 \cdot 3^2 = 72$). The third approach for finding the LCM is the Build-up Method where after expressing 24 and 36 in their prime factor exponential forms (e.g., $24 = 2^3 \cdot 3^1$ and $36 = 2^2 \cdot 3^2$), the LCM is obtained by building up the prime factors to the larger exponents (e.g., because $2^2 \cdot 3^2$ has more threes than $2^3 \cdot 3^1$, we build up from $24 = 2^3 \cdot 3^1$ to have the same number of threes as $2^2 \cdot 3^2$, making the LCM $2^3 \cdot 3^2 = 72$). The fourth approach for finding the LCM is using the Theorem Method which states that the product of two numbers is equal to the product of their GCF and LCM (e.g., because the GCF of 24 and 36 is 12, LCM of 72 is obtained by dividing 24×36 by 12).

After the instruction, the pre-service teachers were evaluated by means of a quiz and a survey. In a quiz of 12 problems, problems 1, 2, 3, and 4 involved finding the GCF of 45 and 75 using the first, second, third, and fourth approaches, respectively. Problems 5, 6, 7, and 8 involved finding the LCM of 45 and 75 using the first, second, third, and fourth approaches, respectively. Problem 9 and 10 involved finding the GCF and LCM of 12 and 18 using any method. Problem 11 and 12 involved finding the GCF and LCM of 2,873 and 3,757 using any method. Each problem in the quiz was scored as 1 if the correct answer was supported by clear explanations and logical arguments; otherwise, it was scored as 0. Thus, the quiz score ranged from 0 to 12. In the survey of two questionnaires, they were asked about their preference of finding the GCF and LCM based on their problem solving experience. They were also required to write one or two paragraphs explaining any criteria they identified for their choices of preferred approaches, as well as providing a comparison and contrast analysis of the different approaches for finding the GCF and LCM.

FINDINGS

Based on the survey, the first, second, third, and fourth approaches of finding the GCF were preferred by 2, 7, 6, and 22 pre-service teachers, respectively. The first, second, third, and fourth approaches of finding the LCM were preferred by 3, 20, 5, and 9 pre-service teachers, respectively.

Although the majority of the pre-service teachers recognized that the Set Intersection Method for finding the GCF and LCM was "clunky" and "only works for small numbers," they agreed that such method was conceptually the more "natural" way of making sense of the GCF and LCM. The Prime Factorization Method for finding the GCF and LCM was the more "familiar" approach that most pre-service teachers "learned in grade school." The pre-service teachers considered the Euclidian Algorithm Method the most efficient approach for finding the GCF because it "works for any numbers, including large ones" and "simplifies the steps in the Repeated Subtractions Method." On the other hand, the Build-up Method was not favourable because it was viewed as less efficient than the Prime Factorization Method. While the Theorem Method was not the most popular approach, those who preferred it said it was the most efficient and "easiest" approach "if you figure out the GCF beforehand, especially for big numbers."

Supposing that the Euclidian Algorithm Method and the Theorem Method were the most efficient approaches for finding the GCF and LCM, respectively, as the pre-service teachers assessed in general, it was apparent that those who preferred either of those two approaches performed well above those who preferred other approaches. The average scores of all problems of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the GCF were 4, 6.7, 6.6, and 9.4, respectively. The average scores of all problems of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the LCM were 8, 7.6, 8, and 11, respectively.

In relation to their understanding of the most efficient approaches for finding the GCF and LCM, the pre-service teachers' performance on problems 4 and 8 (problems involving the most efficient methods for the GCF and LCM, respectively) was highly indicative of their likelihood of preferring those most efficient approaches. The average scores of problem 4 of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the GCF were 0, 0.1, 0.1, and 0.8, respectively. The average scores of problem 8 of the pre-service teachers who preferred the first, second, third, and fourth approaches of finding the LCM were 0.3, 0.1, 0.2, and 0.9, respectively.

Ten of the 12 pre-service teachers who successfully solved problems 11 and 12 (problems involving finding the GCF and LCM of larger numbers) solved both problems using and chose as their preferred approach the Euclidian Algorithm Method or the Theorem Method. Only twelve of the 25 pre-service teachers who successfully solved problems 9 and 10 (problems involving finding the GCF and LCM of smaller numbers) but not problems 11 and 12 chose as their preferred approach the Euclidian Algorithm Method or the Theorem Method. In other words, the more-mathematically-able pre-service teachers were about twice as likely, in proportion to their group membership, both to solve them using and to prefer the Euclidian Algorithm Method or the Theorem Method to other approaches as the less-mathematically-able pre-service teachers. To this extent, the pre-service teachers' mathematics content knowledge was a determining factor in their appreciation for the most efficient approach.

Nevertheless, like the majority (23) of the 25 pre-service teachers who successfully solved problems 9 and 10 but not problems 11 and 12, 11 of the 12 pre-service teachers who successfully solved problems 11 and 12 was more likely to solve problems 9 and 10 (problems involving finding the GCF and LCM of smaller numbers) using either the Set Intersection Method or the Prime Factorization Method than any other approaches. Evidently, the more-mathematically-able pre-service teachers appeared to be more flexible in choosing problem solving strategy, depending on the level of difficulty of the problems, in particular, the magnitude of the numbers involved in the problems.

This, to some extent, demonstrated, from a point of view of number theory, a similar notion of the "apparently counter-intuitive inverted aptitude-strategy relationship" based on the findings by Roberts, Gilmore, and Wood (1997): the pre-service teachers who were more fluent in the more sophisticated approaches for finding the GCF and LCM (e.g., the Euclidian Algorithm Method and the Theorem Method) ingeniously avoided the use of such sophisticated approaches when solving simpler problems. One explanation to this flexibility might be that these more-mathematically-able pre-service teachers consciously attended to one attribute of an elegant approach, namely, simplicity (Silver & Metzger, 1989): simpler problems could and should be solved using a more elementary strategy (e.g., the Set Intersection Method or the Prime Factorization Method), instead of a more advanced strategy (e.g., the Euclidian Algorithm Method or the Theorem Method).

Indeed, some of them explained that "I only began to see why we need to learn many different approaches until you gave us the last four problems [problems 9, 10, 11, and 12] all at once," while others deduced that "some methods work for some problems, while other methods work better for other problems." Their explanations suggested, to some extent, that they valued the need to study more than one approach in order to better appreciate other approaches. It was clear that the more approaches they understood, the more positive they were towards the practice of problem solving using many different approaches, and thus, the more mathematically mature they became to appreciate the various characteristics of a mathematically "beautiful" approach, including the idea of efficiency and simplicity.

Such flexibility in adapting alternative approaches was yet not observed in the eight pre-service teachers who successfully solved problems 9 and 10 but not problems 11 and 12. These less-mathematically-able pre-service teachers persisted in applying either the Set Intersection Method or the Prime Factorization Method to solve problems 11 and 12, albeit unsuccessfully. While the Set Intersection Method could be viewed as the more "natural" way of conceptualizing the GCF and LCM, this evidence suggested, to some extent, that such approach was realized by these less-mathematically-able pre-service teachers more at the procedural level, rather than at the conceptual level.

CONCLUSIONS AND DISCUSSIONS

The current study explored the relationship between pre-service teachers' mathematics content knowledge and their preferred approaches for finding the GCF and LCM, as well as their predisposition to favour mathematically "beautiful" approach. Two major findings were observed. First, the more-mathematically-able pre-service teachers were more likely than the less-mathematically-able pre-service teachers to recognize the most efficient approach. An aesthetic appreciation for the most efficient approach appeared to necessitate a certain level of understanding of that approach; to some extent, it was not possible to appreciate the most efficient approach if one lacked the understanding of problem solving using many different approaches. Second, the more-mathematically-able pre-service teachers were more likely than the less-mathematically-able pre-service teachers to adaptively vary their problem solving strategies to accommodate the level of difficulty of the problems. The more tools they could work with to solve a problem, the more options they had when reflecting to decide which tool would be appropriate for which situation.

Two pedagogical recommendations might be proposed. First, mathematics learning experience, perhaps as early as the elementary school level, could involve problem solving using many different approaches. Given sufficient exposure to a variety of different methods to solve the same problem involving the same mathematics concept, beginning problem solvers might become not only fluent in many different problem solving approaches but also creative in looking for novel problem solving approaches and flexible to recognize the appropriateness of utilizing certain problem solving approaches in solving particular situations. Second, aesthetic appreciations towards mathematical "beauty" could be nurtured to young children, even if they might only concern with the idea of efficiency in terms of time and the number of steps to solve a problem. To this end, mathematics teachers could promote classroom discussions that required students to compare and contrast different problem solving approaches.

References

Bodemer, D., Plötzner, R., Feuerlein, I., & Spada, H. (2004). The active integration of information during learning with dynamic and interactive visualizations. *Learning and Instruction*, *14*, 325-341.

De Jong, T., Ainsworth, S., Dobson, M., van der Hulst, A., Levonen, J., Reimann, P., … Swaak, J. (1998). Acquiring knowledge in science and mathematics: The use of multiple representations in technology-based learning environments. In M. van Someren, P. Reimann, H. Boshuizen, & T. de Jong (Eds.), *Learning with multiple representations* (pp. 9-41). Oxford, England: Elsevier Sciences.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetic of mathematical thought. *For the Learning of Mathematics*, *6*, 2-10.

Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics*, *66*, 349-371.

Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Educational Studies in Mathematics*, *52*, 151-176.

Reeves, L. M., & Weisberg, R. W. (1994). The role of content and abstract information in analogical transfer. *Psychological Bulletin*, *115*, 381-400.

Roberts, M. J., Gilmore, D. J., & Wood, D. J. (1997). Individual differences and strategy selection in reasoning. *British Journal of Psychology*, *88*, 473-492.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Font Strawhun, B. T. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior*, *24*, 287-301.

Silver, E. A., Leung, S. S., & Cai, J. (1995). Generating multiple solutions for a problem: A comparison of the responses of U.S. and Japanese students. *Educational Studies in Mathematics*, *28*, 35-54.

Silver, E. A., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D. McLeod, & V. Adams (Eds.), *Affect and mathematical problem solving* (pp. 59-74). New York: Springer-Verlag.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6,* 261-284.

Skemp, R. (1987). *The psychology of learning mathematics*. Mahwah, NJ: Erlbaum.

Tabachneck, H. J., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense-making in mathematics problem solving. In A. Ram & K. Eiselt (Eds.), *Proceedings of the sixteenth annual conference of the Cognitive Science Society* (pp. 836-841). Hillsdale, NJ: Erlbaum.

Van Someren, M. W., Boshuizen, H. P., De Jong, T., & Reimann, P. (1998). Introduction. In M. van Someren, P. Reimann, H. Boshuizen, & T. de Jong (Eds.), *Learning with multiple representations* (pp. 1-5). Oxford, England: Elsevier Sciences.