# Why is there no quintic formula?

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# Abel–Ruffini theorem

The theorem, also known as *Abel's impossible theorem*, says following

**Theorem**

*There is no closed formula solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.*

A closed formula solution to an equation is a formula of the coefficients only involving additions, subtractions, multiplications, divisions and root extractions such that it always gives a root to the original equation.

# Examples and nonexamples

Examples are

- Quadratic formula: $x = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$.
- Cubic formula: $x = \sqrt[3]{q + \sqrt{q^2 + (r - p^2)^3}} + \sqrt[3]{q - \sqrt{q^2 + (r - p^2)^3}} + p$, where $p = -a/3, r = b/3$ and $q = p^3 + ab/6 - c/2$.
- Quartic formula: a very complicated formula involving several 3-nested root extractions, which this slide is too narrow to contain.

For a non-example, it is obvious that $x = a$ is a root to the quintic equation $x^5 - a^5 = 0$ for any $a$, but this is *NOT* the closed formula we want to discuss today.

# Tom and Jerry

Your friend Tom, claims that he has a closed formula that solves all quintic equations $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$. Below is his formula:

$$x = \frac{1}{a^5 + \frac{b^4 - cda}{a + bc - de + 1} - \frac{a - \frac{c^2}{d+e}}{a^6 - 1}}$$

How do you, smart Jerry, refute him?

Figure: Tom and you

# Direct Approach

A direct approach is to find a particular equation, plug in the coefficients into Tom's formula, and verify that the resulf of Tom's formula is not a root.

Great! Let us take

$a = 100, b = \pi, c = 5000, d = -0.001, e = \mathrm{e} = 2.718281828...$, and plug into Tom's formula

$$x = \frac{1}{a^5 + \frac{b^4 - cda}{a + bc - de + 1} - \frac{a - \frac{c^2}{d+e}}{a^6 - 1}}$$

I don't want to do the calculation!

## Better Approach

Let us denote all five roots by $x_1, x_2, \cdots, x_5$. Then we have

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

Expanding both sides gives us Vieta's formula,

**Theorem (Vieta's Formula)**

*If $x_1, x_2, \ldots, x_5$ are roots to the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e$, then*

$$\begin{aligned} -a &= x_1 + x_2 + \cdots + x_5 \\ b &= x_1x_2 + x_1x_3 + \cdots + x_4x_5 \\ -c &= x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5 \\ d &= x_1x_2x_3x_4 + \cdots + x_2x_3x_4x_5 \\ -e &= x_1x_2\cdots x_5 \end{aligned}$$

So each coefficient is a *symmetric* function of these five roots.

# Better Approach

Let us view all five roots $x_1, \cdots, x_5$ as five points on the complex plane. If we let $x_1$ move to $x_2$, $x_2$ move to $x_3$, ..., $x_5$ move to $x_1$, then $a, b, c, d, e$ will move around. Because they are functions of $x_1, \cdots, x_5$.
But eventually they will come back to where they are. Because they are symmetric in $x_1, \cdots, x_5$. For example, if we make the change $x_1 \to x_2, x_2 \to x_3, \cdots, x_4 \to x_5, x_5 \to x_1$, then $a = -x_1 - \cdots - x_5$ will stay the same when the movement is done.

# Better Approach

Assume that Tom is right. Then his formula must be one of the root, say $x_1$. If we permutate roots around as mentioned earlier, then

- on one hand, $x_1$ becomes $x_2$, so Tom's formula must give us $x_2$ after all movements;
- on the other hand, all coefficients, as functions of roots, are unchanged, therefore Tom's formula still gives us $x_1$.

$$x = \frac{1}{a^5 + \dfrac{b^4 - cda}{a + bc - de + 1} - \dfrac{a - \frac{c^2}{d+e}}{a^6 - 1}}$$

# Better Approach

Therefore Tom's formula must be wrong!
The talk is over! Thanks!

# Wait a second!

Where did we use the fact that it is a quintic equation? It seems that this argument works for any equation! So did we prove that any equation with at least two roots has no closed formula solutions?

Of course *NOT*! We all know that there is quadratic formula!

Let us go back and look at the quadratic case $x^2 + ax + b = 0$. We know that

$$x_{1,2} = \frac{-a \pm \sqrt{\Delta}}{2}$$

where $\Delta = a^2 - 4b$. Using Vieta's formula, we can rewrite it as $\Delta = (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2$.

# Quadratic Case

What happens when we switch $x_1$ and $x_2$? Of course $a = -x_1 - x_2$ and $b = x_1x_2$ remain unchanged, so is $\Delta$. But how about $\sqrt{\Delta}$?

From the movie, we see that $\sqrt{\Delta}$ actually changes its value when we switch $x_1$ and $x_2$! Therefore the quadratic formula also changes from $x_1$ to $x_2$.

# Summary I

So the better approach suggests that any formula does not involve root extractions, does *NOT* give a solution to our equation. Because no root extractions means no change in outputs when we move roots around, which is a contradiction!

However, your friend, Tom, shamelessly changed his formula to following

$$x = \sqrt{a + \sqrt[3]{b^4 + cd}} - \sqrt[5]{e + \sqrt{c^2 - ab}}$$

# Some Terminology

- “Moving roots around” is called a *permutation* in group theory, which, in our case, is an element of $S_5$, the *symmetric group* on 5 objects.
- The number of rounds, that a complex valued function $z = f(t)$ winds around the origin in the complex plane as $t$ changes, is called the *winding number* of $z$.

# Ultimate Approach

How do we refute Tom again? The key point is that root extraction detects the change of winding numbers!
So we want to find a permutation in $S_5$ such that

- it is not the identity element, i.e., it moves Tom's solution $x_1$ to a different root, e.g., $x_2$;
- the winding number of each coefficient induced by the action of this element is zero.

Therefore the same argument works: on the one hand, zero winding number implies no change in outputs; on the other hand, we've moved $x_1$ to a different solution. So Tom's formula should give us a new solution, where we reach a contradiction!

# Weapon: commutator

### Fact

If some permutation $\sigma$ moves $z$ around the origin by an angle $\theta$, then $\sigma^{-1}$ moves $z$ around the origin by $-\theta$.

A *commutator* is an element of the form $\sigma\tau\sigma^{-1}\tau^{-1}$, denoted by $[\sigma, \tau]$. By the fact above, a commutator clearly does not change the winding number of each coefficient. And thanks to the fact that $S_5$ is *NOT* abelian, a commutator in $S_5$ is not always the identity element.
If we have a formula, e.g., $\sqrt[26]{a + b^2 - dc^6 + e^{12343}}$, no matter how crazy the formula inside the radical is, the winding number of that part is zero under the action of a commutator. Therefore the commutator does not change the output.

# One More Thing

Let us go back to Tom's formula

$$x = \sqrt{a + \sqrt[3]{b^4 + cd}} - \sqrt[5]{e + \sqrt{c^2 - ab}}$$

We know that a commutator will not change $\sqrt[3]{b^4 + cd}$, it of course does not change $a$. But how about the winding number of their sum? We need that to be zero in order to get a contradiction!
The way to get around this issue is to use a commutator of commutators, i.e., to use $[\sigma, \tau]$, where $\sigma = [\sigma_1, \tau_1]$ and $\tau = [\sigma_2, \tau_2]$ are both commutators. Both $\sigma$ and $\tau$ will fix whatever inside the radical sign, and their commutator will fix the whole expression.

# Summary II

So in order to refute Tom's formula, which involves a nested radical expression, we have to find a permutation in $S_5$ such that

- it does not fix $x_1$, e.g., it moves $x_1$ to $x_2$;
- it is a commutator of commutators in $S_5$.

Is this possible?

# Finally, some calculation

Let us do some calculation by .... hand? by a computer!

```
In [1]: S5 = SymmetricGroup(5)

In [2]: A5 = S5.commutator()

In [3]: A5.order()

Out[3]: 60

In [4]: B = A5.commutator()

In [5]: B.order()

Out[5]: 60
```

Figure: SageMath

# Finally, some calculation

From the calculation, we notice that the commutator subgroup $B$ of $A_5$ is still $A_5$! So each element in $A_5$ is of the form $[\sigma, \tau]$ for some $\sigma, \tau \in A_5$.
Since $\sigma$ and $\tau$ are both in $A_5$, they are also $[\sigma', \tau']$ and $[\sigma'', \tau'']$ respectively. Then, since $\sigma', \sigma'', \tau', \tau''$ are all elements in $A_5$, ...
So each element in $A_5$ is a commutator of commutators of commutators of ... of commutators of commutators.
So no matter how many times of nested radical expression appears in Tom's formula, we can always find some non-identity element in $A_5$ such that the winding number of each radical expression of the formula is zero under its action. Then the output is unchanged, but $x_1, \ldots, x_5$ are secretly permutated. Therefore Tom's formula cannot be legitimate.

# Yeah!

SO WE WIN!

# Solvable Groups, Galois Theory and more ...

In modern mathematics, the materials we "discussed" today correspond to the fact that $S_5$ is *NOT* solvable. All topics here are part of a beautiful math theory called "Galois Theory". It establishes a connection between field theory and group theory.

The "proof" I presented today is due to Vladimir Arnold in 1963. This is part of the topological Galois theory– a theory parallel to Galois theory, but deals with fundamental groups of topological spaces and their covering spaces.

You are strongly encouraged to learn more about all of these.

# Byproducts

We also have some byproducts for cubic and quartic equations.
It turns out that

- the commutator subgroup $S_3^{(1)}$ of $S_3$ has 3 elements and the 2nd commutator subgroup $S_3^{(2)}$ is trivial. So any formula does not involve a nested root extraction is necessarily *NOT* a cubic formula!
- the commutator subgroup $S_4^{(1)}$ of $S_4$ has 12 elements, the 2nd commutator group $S_4^{(2)}$ has 4 elements and the 3rd one $S_4^{(3)}$ is trivial. So any formula does not involve a 3-nested root extraction is necessarily *NOT* a quartic formula!

Let us go back and check our conclusion. go to slide

# Thank you!

Thanks for listening!