Why the AMC's are Trivial

Brandon Jiang

January 24, 2016

1 How to Use this Document

This could possibly be used as a sort of study guide, but its main intent is to offer students some direction to prepare for this contest other than just doing past problems. Note that it is assumed that the reader is mathematically capable of understanding the standard curriculum at school. If not, the author recommends a different hobby such as biology or debate. Painstakingly crafted, this will make all (most) future problems encountered by the reader trivial. Enjoy!

2 Algebra

It should be noted that most problems in this contest are algebra based, meaning that most solutions will require some use of algebra. Besides the topics below, one should have a solid grasp on basic algebraic manipulation and solving equations. In addition, the author would like to comment that contest algebra is the source of the most contrived problems on the test.

2.1 Manipulation

2.1.1 Identities

One should definitely be aware of the following identities:

- $a^2 - b^2 = (a+b)(a-b)$ (difference of squares)
- $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$ (squares of sums)
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$ (square of sum of 3 variables)
- $a^3+b^3 = (a+b)(a^2-ab+b^2)$ and $a^3-b^3 = (a-b)(a^2+ab+b^2)$ (difference of cubes)
- $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

If he or she wishes, the curious student may ask the author on factoring techniques for contrived multivariable polynomials such as the last one in the list.

2.1.2 Substitutions

There is little that can be taught about these, except that if you chose the right one the result is typically very powerful. The author's best advice is to choose one that best fits the problem; for example if you have many factors of $x-5$ in an expression consider making the substitution $y = x - 5$.

- Linear substitutions are generally useful in this manner; it should be noted that substituting in order to arrive at symmetric expressions tends to be useful as well.

- A certain substitution that problem writers enjoy is what the author has aptly named the Vieta substitution: Generally in two variable expressions with, say, a and b one may find the substitution $u = a + b$ and $v = ab$ very useful, it should be noted that with this definition we also have that a and b are the roots of the quadratic $x^2 - ux + v$.

2.2 Functions

Know the basic definitions of functions; most notably a function may not have two different outputs for a single input. (Vertical line test)

2.2.1 Polynomials

- Make sure you at least know what a polynomial is. There is bound to be a couple questions related to polynomials on the exam.

- Be familiar with the Binomial Theorem, which states that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

  Also be able to recognize coefficients of a polynomial Pascal's triangle so you can manipulate to make a perfect nth power polynomial.

- Know Vieta's relations regarding symmetric sums of the roots of a polynomial. For an nth degree polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real coefficients, roots $r_1, r_2, \cdots, r_n$ and $a_n \neq 0$,

$$-\frac{a_{n-1}}{a_n} = r_1 + r_2 + \cdots + r_n,$$

$$\frac{a_{n-2}}{a_n} = r_1 r_2 + r_2 r_3 + \cdots + r_n r_1,$$

$$\cdots$$

$$(-1)^n \frac{a_0}{a_n} = r_1 r_2 \cdots r_n.$$

  The first and last (sum and product, respectively) are by far the most useful.

- Be able to factor polynomials that are intended to be factored. (Generally this means you can easily guess its roots using the Rational Root Theorem).

- Know that r is a root of $P(x)$ if and only if $P(r) = 0$; a corollary of this is that the remainder when $P(x)$ is divided by $x - a$ is $P(a)$.

- It may be helpful to write a polynomial in it's factored form, or

$$P(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

- Know that $P(1)$ gives the sum of the coefficients of $P(x)$.

- Be aware of complex roots and basic operations with complex numbers, and consider learning both polar and exponential form.

- Consider learning about the roots of unity, or the roots of $x^n - 1$. Note that 1 is always a factor and roots of unity should come to mind when someone sees an expression like $1 + x + x^2 + \cdots + x^n$ since $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$.

2.2.2 Other functions

- Be familiar with the six trigonometric functions and their properties, most notably which ones are even/odd, Pythagorean identities, and sum and difference identities.

- Know the basic properties of exponents and logarithms (sum, product, change-of-base, etc.)

- A certain function that comes up is the floor or ceiling function, be aware of their definition. Note that it is a common technique to write $\lfloor x \rfloor = x - \{x\}$ where $\{x\}$ denotes the fractional part of x. (It is between 0 and 1). Make use of the fact that $\lfloor x \rfloor \leq x$ and that the floor function always returns an integer.

- The absolute value function also may appear, just be aware of how it works and know the generalized triangle inequality, which states that for complex numbers a_i,

$$\sum_{i=1}^{n} |a_i| \geq \left| \sum_{i=1}^{n} a_i \right|.$$

- For piece-wise defined functions, be sure to observe function behavior on borders.

- Sometimes, there will be a problem with very sophisticated looking notation but in reality is not too difficult. Be able to overcome notation on some of these weird functions.

- Be on the lookout for cyclic functions. If you see a rational function, there is a chance it will be cyclic, thus making things a lot easier. (This means that for some positive integer n, $f^n(x) = f(x)$).

2.3 Sequences

- Know that arithmetic sequences have a common difference and geometric sequences have a common ratio.
- Know formulas for the sums of sequences (series).
- When asked to evaluate a complicated sum or product, list a few terms to see if cancelation of terms occurs. Be aware of telescoping sequences. Most prominent is

$$\sum_{n=1}^{m} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1}$$

  Note that all the middle terms cancel, leaving us with $1 - \frac{1}{m+1} = \frac{m}{m+1}$. Sometimes factoring then decomposing an expression like this will allow you to collapse the entire thing like above. Be creative!

- Know the following series:
  - $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ (triangular numbers)
  - $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$
  - $1 + 3 + 5 + \cdots + 2n - 1 = n^2$
  - $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
  - $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

3 Geometry

For some reason, both the author and contest writers love geometry, even though nearly all Euclidian geometry is trivial with coordinates (contingent on computational prowess and time). Geometry tends to be forgotten among typical students due to the fact that the public school curriculum only offers one year of basic geometry. If you wish to do well on this contest, it is imperative that you get decent at geometry. The author stresses the importance of drawing a diagram when solving most geometry problems.

3.1 Similarity

- If two objects are similar, then their respective length ratios are equivalent and all angles are the same. The most common way to show this is with two equal angles.
- The technique of finding angles in a problem is called angle chasing, which is probably the most useful skill in contest geometry.
- Angle chasing generally comes down to alternate interior angles, parallel lines, and angles in a triangle summing to 180°.
- Sometimes similar triangles are not given, and you will have to make them yourselves. The most common way (accompanying angle chasing) to do this is dropping perpendiculars or by drawing parallel lines.
- Continuing from above, always make note of right angles, since they usually make things a lot easier.

3.2 Circles

A circle is the locus of points equidistant from its center. These figures are quite prominent in geometry, and are extremely useful for angle chasing. Common techniques are to draw radii to tangents, draw common tangents, and draw the segment between the centers of the circles.

- Area of a circle: πr^2.
- Circumference of a circle: $2\pi r$.

3.2.1 Useful Angle-Chasing Facts:

- Central angles, with the vertex at the center, have their measure equal to the measure of the arc.
- Inscribed angles equal to half the measure of the arc, and as a result inscribed angles subtending to the same arc are equal. Be aware of the case when one ray is tangent to the circle.
- As a result, inscribed angles with two points diametrically opposite each other have a measure of 90°.
- The angle formed between two secants when the vertex lies outside the circle is half the difference of the arcs subtended.
- The angle formed between two chords when the vertex lies within the circle is half the sum of the two arcs subtended.

3.2.2 Power of a Point

These formulas are almost guaranteed in a circle problem, and provide one with valuable information on lengths.

- Let the point P be the intersection of two chords of the circle, say AB and CD. Then,
$$(AP)(PB) = (CP)(PD).$$
- Let P be the intersection of two secants in a circle, say AB and CD, and let PE be the tangent from P to the circle at E. Then, we have that
$$(PA)(PB) = (PC)(PD) = (PE)^2.$$
- These formulas generalize to describe the "power" of a point respect to a circle as a function. This is useful because we know the product of any two of those lengths once we know the power.
- Between two circles, the radical axis is the line on which all points have the same power respect to both circles. For three circles, the radical axes concur at a single point called the radical center.

3.3 Triangles

Hopefully you know what a triangle is. Most contest problems rely on this most basic polygon of three sides. Sadly, this document will not provide enough information for a certain degree of proficiency regarding triangles; nothing can replace doing practice problems. Below are very useful results regarding triangles with respect to the arbitrary triangle $\triangle ABC$ with angles $\angle A, \angle B$ and $\angle C$, and side lengths (opposite the respective angle) a, b, and c:

- The angles in a triangle sum to $180°$.
- Lines passing through any vertex of a triangle are called cevians. It is trivial to show that they will intersect exactly one side of the triangle (you may have to extend sides).
- A median is a cevian intersecting the opposite side at its midpoint. If all three medians are drawn, they concur at the centroid of the triangle. It is well known that medians divide each other in a $2:1$ ratio.
- An angle bisector is a cevian which bisects the angle at the vertex it passes through. The three angle bisectors concur at the incenter of the triangle (the circle within the triangle that is tangent to all three side lengths. (Note: The author is aware that each vertex has two angle bisectors, and taking that third case gives the result of the three excenters).
- An altitude is a cevian which is perpendicular to the side opposite the vertex. This is especially useful with all the right angles, and the three altitudes concur at the orthocenter.

- A useful tool for proving concurrency is Ceva's Theorem, which states that for points D, E, and F on sides BC, CA, and AB, lines AD, BE, and CF are concurrent if and only if

$$\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1.$$

To answer the question of the inquisitive reader, this theorem can prove all the above results (but it is not the only method).

- The perpendicular bisectors of each side in a triangle concur at the circumcenter (try proving this), or the center of the circle that passes through points A, B and C.

- Here is a slew of triangle formulas, ranked in order of importance (by the author):

  - Area of a triangle:
    * $[ABC] = \frac{1}{2}BC \cdot h_A = \frac{1}{2}CA \cdot h_B = \frac{1}{2}AB \cdot h_C$
    * $[ABC] = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$
    * $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$
    * $[ABC] = rs$, where r is the inradius.
    * $[ABC] = \frac{abc}{4R}$, where R is the circumradius.
  - The Pythagorean Theorem states that for a right triangle with legs a and b and hypotenuse c,

    $$a^2 + b^2 = c^2.$$

    This formula probably gets the most abuse by problem writers. Make sure you also know most primitive Pythagorean triples.
  - The Triangle Inequality states that for any nondegenerate triangle,

    $$a + b > c, b + c > a, c + a > b$$

  - The Angle Bisector Theorem states that if the angle bisector of $\angle A$ meets BC at D, then

    $$\frac{AB}{AC} = \frac{BD}{CD}.$$

  - The Ratio Lemma is a generalization of the angle bisector theorem and states that for any D on BC,

    $$\frac{AB}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle CAD} = \frac{BD}{CD}.$$

  - The Law of Cosines is a generalization of the Pythagorean theorem and states that

    $$c^2 = a^2 + b^2 - 2ab\cos C.$$

– The Extended Law of Sines states that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{b}{\sin C} = 2R$$

– Stewart's Theorem states that if D is a point on BC such that $AD = d$, $BD = m$, and $CD = n$, then

$$man + dad = bmb + cnc.$$

(Or, as the author likes to say, a man and his dad put the bomb in the sink.)

3.4 (Mostly Cyclic) Quadrilaterals and Other Polygons

Know the generic quadrilaterals and their properties (parallelogram, trapezoid, kite, rhombus, etc.) Typically, nothing extremely dependent on definition comes up on the exam, so that should be the least of your worries.

3.4.1 Cyclic Quadrilaterals

A cyclic quadrilateral is one that can be inscribed in a circle. As one would expect, this has some powerful implications. For a cyclic quadrilateral $ABCD$,

- $\angle ABC + \angle CDA = \angle BCD + \angle DAB = 180°$.
- $\angle ABD = \angle ACD$, $\angle BCA = \angle BDA, \angle CDB = \angle CAB$ and $\angle DAC = \angle DBC$. All of this follows directly from inscribed angles.
- The area of a cyclic quadrilateral is $\sqrt{s(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{a+b+c+d}{2}$.
- Be aware that you can apply power of a point on lengths relating to a cyclic quadrilateral.
- Ptolemy's theorem states that for a cyclic quadrilateral,

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

3.4.2 Other Quadrilaterals and Polygons

Be able to use properties of other quadrilaterals to solve problems as well. Most common are the trapezoid and the parallelogram.

- Area of trapezoid: $\frac{1}{2}(b_1 + b_2)h$.
- Area of parallelogram: bh.
- Square and rectangles are special cases of parallelograms; be aware of certain angle and side conditions that result.

- Area of a rhombus: $\frac{1}{2}d_1 d_2$, where d_1 and d_2 are the lengths of its diagonals.
- For problems involving quadrilaterals, also be on the lookout for similar triangles as well.
- The sum of the angles in an n-gon is $180(n-2)$.
- The number of diagonals in an n-gon is $\frac{n(n-3)}{2}$.
- Polygon problems generally break down into triangles and quadrilaterals, so be aware of this.
- If a polygon has an inradius, then its area is rs where r is its inradius and s is its semiperimeter.

3.5 3-D Geometry

- Know the generic surface area and volume formulas of geometric figures.
- Volume of rectangular prism: $V = lwh$.
- Surface area of rectangular prism: $A = 2(lw + wh + hl)$.
- Space diagonal of rectangular prism: $d = \sqrt{w^2 + h^2 + l^2}$.
- Volume of sphere: $V = \frac{4}{3}\pi r^3$.
- Surface area of sphere: $A = 4\pi r^2$.
- Volume of cylinder: $V = \pi r^2 h$.
- Surface area of cylinder: $V = 2\pi r^2 + 2\pi rh$.
- Volume of cone: $V = \frac{1}{3}\pi r^2 h$.
- Surface area of cone: $A = \pi r^2 + \pi rl$, where $l = \sqrt{r^2 + h^2}$.
- 3-D geometry problems generally break down into taking the 2-D cross section; for example if one inscribes a cylinder in a sphere the cross section becomes a triangle inscribed in a circle. Beware of tricky slant heights!
- There are exactly five regular polyhedra: cube, tetrahedron, octahedron, dodecahedron, and icosahedron. Know basic properties of the first three (the next two don't really come up).
- Euler's theorem tells us that for any polyhedron,

  $$V - E + F = 2,$$

  where V, E, and F are the vertices, edges, and faces, respectively.

4 Number Theory

Number theory is quite similar to algebra but works with mainly the positive integers. This is also problematic for those new to contests because number theory is removed from the curriculum by the time we reach 5th grade. Many of these results are very direct in solving contest problems. Typically, word problems involving discrete amounts is a hint that it is a number theory problem.

4.1 Primes and Divisibility

- Make sure you know the divisibility rules for all integers under 11. When given an equation, you may find it helpful to verify that both sides of the equation are divisible by the same numbers.
- It often helps to write out a number in terms of its prime factorization.
- The number of factors a number n with $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, such that $p_1, \cdots, p_k$ are primes is
$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$
- If a number n has d factors, then the product of its factors is
$$n^{\frac{d}{2}}.$$
- Given a number n with the above prime factorization, the sum of its factors is:
$$(p_1^0 + p_1^1 + \cdots + p_1^{e_1})(p_2^0 + p_2^1 + \cdots + p_2^{e_2}) \cdots (p_k^0 + p_k^1 + \cdots + p_k^{e_k}).$$
- It is an interesting fact that if $s(n)$ denotes the sum of the digits of n, then $n \equiv s(n) \pmod 9$.
- The Euclidian Algorithm allows us to simplify the greatest common denominator of two numbers, by subtracting multiples of the other number from the first number. For example $\gcd(21, 112) = \gcd(21, 112 - 21 \cdot 5) = \gcd(21, 7) = \gcd(0, 7) = 7$. This technique is especially effective on expressions with variables as well.

4.2 Modular Arithmetic

There exists a very nice handout on modular arithmetic by the author. This is a very useful tool in number theory that focuses on the remainders of each number. Many times you will typically encounter problems asking for the last digit (dividing by 10), the last two digits (dividing by 100), or the last three digits (dividing by 1000).

- Be familiar with basic properties in modular arithmetic; that is, addition and multiplication of two expressions hold but division does not always work.

- When you have large bases of exponents, you should always reduce the base $\pmod{n}$ first. For example, if we wish to find $16^5 \pmod{7}$, we can simplify the base to $2^5 \pmod{7}$ instead.

- In addition, know that if $\gcd(m, n) = 1$, then m has a modular inverse $\pmod{n}$.

- Know how to solve basic linear and quadratic congruences.

- Sometimes it may help to rewrite the congruence $a \equiv b \pmod{p}$ as $a = b + nk$ for some integer n instead.

- Due to the nature of modular arithmetic itself, chances are a sequence of any pattern involving mods will be cyclic.

- More importantly, the units digit (or any other modulus) of the consecutive powers of a number is cyclic. The reader is encouraged to discover these patterns for him or herself. For example, the units digit of the powers of 3 form the cycle $3, 9, 7, 1, 3, 9, 7, 1, \cdots$.

- Usually there are limited values for what a perfect square, cube, etc. can have $\pmod{p}$. For example, $a^2 \not\equiv 2 \pmod{3}$ and similarly perfect cubes can only be equivalent to $-1, 0, 1 \pmod{9}$.

- It tends to be easier working $\mod p^n$ where p is a prime and ideally $n = 1$. For these purposes, the Chinese Remainder Theorem is useful. For example, one can split a congruence $\pmod{1000}$ into two different congruences $\pmod{8}$ and $\pmod{125}$ and the Chinese Remainder Theorem tells us that knowing both these solutions uniquely determines a solution $\pmod{1000}$.

- Fermat's little theorem states that for any prime p and integer a,

$$a^p \equiv a \pmod{p}.$$

- The Euler totient function, or ϕ function, is a function returning the number of numbers less than and relatively prime to a number. It can be shown with the principle of inclusion and exclusion that for a number $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

  Note that when n is prime, $\phi(n) = n - 1$.

- Euler's Theorem (a generalization of the above) states that for integers a and n such that $\gcd(a, n) = 1$,

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

- The above result is a useful tool in reducing the exponent of a number in an expression. For example, if we wish to find the last three digits of 7^{403}, then we can use the fact that $\phi(1000) = 400$ and reduce it to $7^3 \pmod{1000}$ which gives us 343.

4.3 Diophantine Equations

A Diophantine Equation is an equation in which integer solutions are desired.

- Consider divisibility of both sides of the equation. For example, if one side divides 3 then you know the other side must also divide 3.

- When you see many terms of a certain power (such as 5th powers), taking both equations modulus a small prime or some other number may narrow down you solution set. In fact, sometimes doing this will show that there are no solutions.

- Factoring terms or substituting using techniques in the algebra section may be helpful. Most notably, if you show that x is a multiple of three, consider the substitution $x = 3a$ to possibly discover more about the equation.

- Simon's Favorite Factoring Trick is useful for certain expressions. When you get an expression with an xy, x, and y term, this means that factoring with grouping (SFFT) will suffice.

- Sometimes, consider a complicated multivariable polynomial instead as a single variable polynomial with the other variables as parameters. If this expression is in fact a quadratic, use the quadratic formula. This technique can give us an expression of one variable in terms of another which can useful in narrowing down or finding solutions. For example, if we have $x^2y + 2xy - 5 = 0$, we take it as a quadratic in x and find that

$$x = \frac{-y \pm \sqrt{y^2 + 5y}}{y},$$

  and then narrow down solutions further since $y^2 + 5y$ must be a perfect square.

5 Combinatorics

Combinatorics refers to both counting and probability. For most (but not all students), this is the most difficult subject to understand as well as to consistently solve. However, the author also believes combinatorics to be one of the most interesting of all subjects and without a doubt is the most creative (until standard methods destroy all problems, that is). In addition, there really is no super effective way to learn combinatorics other than through experience. Most combinatorics problems on contests are never seen in school, which may be why the problems seem so difficult at first. This is also the subject where it is easiest to make silly mistakes (off by one or off by a factor of 2).

5.1 Basics

- $n! = n(n-1)\cdots 2\cdot 1$ and $0! = 1$.
- The choose function, $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, returns the number of ways to choose k items out of a set of n items.
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
- $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$
- Be aware of the Pascal's Triangle interpretation of binomial coefficients.
- The Principle of Inclusion and Exclusion for two events states that

$$P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

- For three events,

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C).$$

  (Sometimes it may help to visualize using a Venn diagram.

- Many times when counting a set, we see how many different paths we can take at each step in construction and then multiply the possibilities at each step. While it sounds basic, constructive counting may get difficult to keep track of.
- Be aware whether events are independent or dependent. The number of outcomes in 3 rolls of a die is $6^3 = 216$. However, the number of ways to select a 3 digit number using digits (without repeats) from the set $\{1, 2, 3, 4, 5, 6\}$ is $6 \cdot 5 \cdot 4$ since order matters.

5.2 Bijections

Bijections are possibly the most useful technique in any counting problem, but the difficult part is finding the appropriate one. There are few that appear frequently, but most of the time it is up to the contestant to think of it. Sometimes, these genius bijections may seem to come out of nowhere.

- The number of ways to move from the bottom left corner to the top right corner of a $m \times n$ grid going only up or right is $\binom{m+n}{n} = \binom{m+n}{m}$. In other words, we know we must move right m times and up n times, so our total is identical to the number of ways to order the moves right or up. Variants of this tend to be common, so know how and why this works.

- Complementary counting is a useful bijection as well. For example, say we want to order 5 people given that two people cannot sit together. Then instead of focusing on individual cases, we can first find how to order the people given the two are forced together, where we can treat them like one person. This gives $2 \cdot 4!$ since we can still order the two people within each other. As a result, we subtract from the total to get $5! - 2 \cdot 4! = 72$.

- The number of ways of distributing n indistinguishable items among k different groups is $\binom{n+k-1}{n}$. The way this works is to consider the ordering of n items and $k-1$ "dividers", which then each arrangement gives a unique partition. Note that this also gives the number of nonnegative integer solutions to the Diophantine

  $$x_1 + x_2 + \cdots + x_k = n.$$

  In other ways to apply this, be sure to figure out how what are the items and what are the groups.

- Be able to invoke symmetry in certain problems. As a blatant example, to find the probability that one will get more heads in 2017 flips of a fair coin, we know that there is either more heads or more tails. Clearly, the probability of each is the same, so our answer is $\frac{1}{2}$.

5.3 Probability

For most probability questions, I urge you to use the fact that probability is $\frac{\text{number of desired outcomes}}{\text{number of total outcomes}}$ rather than doing tricky constructive multiplication (which is completely acceptable, but error prone and doesn't always work). This tends to make probability questions the same as counting problems; however, not all problems can be done this way.

- Know the formula for conditional probability, or that $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

- If A and B are independent, then $P(A \cap B) = P(A)P(B)$ and $P(A \cup B) = P(A) + P(B)$.

- Know how to compute expected value. For a random variable X which has value x_i with probability p_i and $\sum x_i = 1$,

  $$E[X] = \sum x_i p_i.$$

- The linearity of expectation is an extremely useful fact about expected value, or that

$$E[X_1 + X_2 + \cdots X_k] = E[X_1] + E[X_2] + \cdots E[X_k].$$

- Note that this means we can disregard whether events are dependent or independent since we only care about expected value.
- Complementary counting (in this case you will subtract from 1) is also useful.

5.4 Recursion

Recursion problems are relatively common on contests, and in general those familiar with computer science tend to be more astute at this. Essentially, recursion allows us to define some function which is rather difficult to count directly, and then we can count a consecutive value in terms of previous values. Note that recursion is not just confined to problems in combinatorics; one may find it useful in other topics such as algebra.

- The most famous recursive sequence is the Fibonacci sequence, defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.
- In general, certain diagrams will have repeated patterns; find out what is added at each step.
- Be able to build one case from a previous case. This is generally the crux of recursion!
- A classic problem states that we are to climb a flight of stairs that is 10 steps long, and each step we take covers either 1 or 2 steps. We are to count how many ways we can go up the stairs. Clearly, other methods are not very effective, so we try recursion. Let $F(n)$ be the number of ways for us to climb a flight of n stairs. It is trivial to find that $F(0) = 1$ and $F(1) = 1$. Now, note that when we are at stair number n, we could have stepped their from either stair $n-1$ (a step size of 1) or from stair $n-2$ (a step size of 2). Thus we have found the (familiar) equation $F(n) = F(n-1) + F(n-2)$, so computing directly we find that $F(10) = 89$.
- Typical recursion problems will be more difficult, but the idea is the same. Make sure you can generalize the construction of each subsequent term, and you will be able to solve the problem.

6 The Actual Exam

The AMC 12 is by no means an easy contest. If you are from the United States, keep in mind that you are from one of the strongest countries in the world in terms of math contests, and that the AMC 12 is the first exam in selecting the national team. However, it is recommended that anyone taking the exam should enter the testing room with confidence in themselves. You should not neglect a problem because it looks hard, nor should you neglect a problem because of its placement on the test. Nevertheless, there are certain strategies that prove extremely effective on the AMC in general.

- One of the most useful techniques that exists is Engineer's Induction. You may be familiar with mathematical induction; this is the engineering analogue. When encountered with a difficult problem, try smaller cases to see if you can find a noticeable pattern. If you can confirm this problem for 3 to 4 cases, then you can probably assume it's true (without rigorous evidence!) and finish the problem. However, note that this is frowned upon and after solving the problem always be sure to look at the intended solution so you can improve (or else you never will).

- For certain geometry problems, you can assume a certain special case and just compute it for that specific one and assume it holds for all. For example, for a geometry question regarding a triangle (without other conditions) you can assume it's equilateral.

- The test is multiple choice! The author encourages guessing under certain circumstances, namely if you have 14 to 15 problems solved, and you really want to make AIME, then you should probably play it safe and leave the rest blank. However, if you are significantly below are above that score, guess away! A recently calculated distribution of answer choices has indicated that B,C, and D occur in the greatest frequency compared to A and E. In practice, I do not believe expected value has a significant effect, and if you do happen to get lucky you will score extremely well. If not...

- Trust your intuition. If you have an innate feeling something is true, skip the rigor and just assume it is. If you do end up with one of the answer choices, there is a high chance you will be correct.

- Time management is quite important on the AMC, since you have an average of 3 minutes per problem. This, of course, assumes that you will actually do every problem. For most beginners, I recommend you solve the first 15 in roughly 30 minutes, and then move on to 16-20. If you have solved the ones you can in the first 20, then move on to the last 5.

- If you are prone to mistakes, I recommend you reserve about 5 minutes to check answers at the end. In addition, use scratch paper freely. If a page starts feeling cluttered, move on. Trees matter but your task at hand matters more (hopefully).

- Most people set their goal at AIME qualification. If this is you, try to get exactly 14 problems correct and leave everything else blank for a score of 100.5.

- If you are reading this and you have scored 110+ on the AMC 12, then it is time for you to start doing harder problems, such as those on the AIME.
- Wow, why did you wait until just now to start preparing? Preparation starts however early you want, and math contests require tons of preparation. If you are in need of resources, a great (free) online resource is the Art of Problem Solving.
- If you have any comments or spot any errors on the handout, feel free to email me at `brandon.r.jiang@gmail.com`. In addition, if you ever need assistance in math or want my commentary on preparation you can ask me as well.

While this document may seem no different from the study guide for one of your classes, keep in mind that math contests transcend the purpose of getting a good score. In the author's opinion, solving contest problems builds intuition and skills in any field and will set you on a path for success. While it is extremely likely you will never have to use the stuff in this handout in the "real world", the entire process of solving difficult problems challenges the brain and hopefully proves enjoyable for you as well.