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CURIOUS MATHEMATICS FOR FUN AND JOY



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THIS MONTH'S PUZZLER

We have that $2^4 = 4^2$. Is this the only example of a pair of distinct positive integers satisfying $a^b = b^a$? Are there rational solutions to this equation?

THE GRAPH OF $y = x^{\overline{x}}$ FOR x > 0.

Using graphing software we see that the

equation $y = x^{\overline{x}}$ produces the graph shown, at least to the right of the vertical axis. (What happens to the left?)



The graph appears to have a maximum value of about 1.4 somewhere near x = 2.7, and decreases towards the value 1 as x grows. It also looks as though it "wants to" adopt the value 0 at x = 0.

With the aid of calculus one can prove that the maximal value of the graph is actually $e^e \approx 1.444$ occurring at $x = e \approx 2.71828$, and that, indeed, $\lim_{x\to 0^+} x^{\overline{x}} = 0$ and $\lim_{x\to\infty} x^{\overline{x}} = 1$. One can also prove that the graph is increasing on the interval (0,e) and decreasing on (e,∞) , as the picture suggests. This means, that for any value M between 1 and e^e , there are precisely two values a < e < b with

$$a^{\frac{1}{a}} = M = b^{\frac{1}{b}}.$$



One checks that $2^{\frac{1}{2}} = 4^{\frac{1}{4}}$ and, of course, that $1^1 = 1^1$, and since 1 and 2 are the only positive integers below e we have that a = 2 and b = 4 give the only distinct

positive integer solution to the equation $a^a = b^b$.

Are there distinct positive rational solutions to $a^{\overline{a}} = b^{\overline{b}}$?

Yes! A wee bit of algebra shows, for any

positive integer
$$n$$
, $a = \left(\frac{n+1}{n}\right)^n$ and $b = \left(\frac{n+1}{n}\right)^{n+1}$ fit the bill.

Comment: Since the first term is clearly less than the second term, we must have that

$$a = \left(1 + \frac{1}{n}\right)^n < e \text{ and } b = \left(1 + \frac{1}{n}\right)^{n+1} > e.$$

In fact, as one learns in calculus class,

 $\left(1+\frac{1}{n}\right)^{n}$ is an increasing sequence

converging to e. (So is $\left(1+\frac{1}{n}\right)^{n+1}$ a

decreasing sequence also converging to e?)

We will prove at the end of this essay that

Any rational solution to $a^{\overline{a}} = b^{\overline{b}}$ with 0 < a < b, must be of the form

$$a = \left(\frac{n+1}{n}\right)^n$$
 and $b = \left(\frac{n+1}{n}\right)^{n+1}$ for some

THE GRAPH OF $x^{y} = y^{x}$ FOR x > 0, y > 0.

If a and b are positive numbers satisfying $a^{\overline{a}} = b^{\overline{b}}$, then "cross exponentiating" gives $a^{b} = b^{a}$. (And conversely, positive reals satisfying $a^{b} = b^{a}$ also satisfy $a^{\overline{a}} = b^{\overline{b}}$.)

As a = 2 and b = 4 are the only distinct positive integer solutions to $a^{\dot{a}} = b^{\dot{b}}$, they are unique distinct positive integer solutions to $a^b = b^a$ as well. This answers the opening puzzler.

The graph of the equation $x^y = y^x$, at least in the first quadrant, has two components: the diagonal line of points with equal coordinate values (x = y) and a curve of points with distinct coordinate values.





As we have seen, for each positive integer n, the point $P_n = \left(\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^{n+1} \right)$ lies on the curve. As n grows, these points approach (e, e), which must thus be the point of intersection of the two components of the graph.

The result we prove at the end of this essay establishes that points P_n and their reflections across the diagonal, are the only points on the graph off the diagonal with rational coordinates. (And the points (2, 4)

and (4,2) are the only off-diagonal points with integer coordinates.)

A CONNECTION TO w^{w^w}

Given a positive real number w, set $a_1 = w$ and $a_{n+1} = w^{a_n}$ for each n > 1. This gives the sequence

$$w, w^{w}, w^{\left(w^{w}\right)}, w^{\left(w^{\left(w^{w^{w}}\right)}\right)}, \ldots$$

This sequence can converge to a finite value (it does for w = 1, for instance) or grow without bound (say, for w = 2).

Comment: When people write $w^{w^{w^w}}$ they mean the sequence w, w^w , $w^{(w^w)}$, To say that $w^{w^{w^w}}$ converges means that the sequence has a finite limit.

Swiss mathematician Leonard Euler (1707-1783) proved that the sequence converges for all values w between $\frac{1}{e^e}$ (which is about 0.066) and $e^{\frac{1}{e}}$, which is the maximum value of the graph $y = x^{\frac{1}{x}}$ (which is about 1.444).

If w is a value between $\frac{1}{e^e}$ and $e^{\frac{1}{e}}$, set M to be the limit value of the sequence. Since $a_{n+1} = w^{a_n}$ and a_{n+1} and a_n each converge to M as n grows, we have

$$M = w^{M}$$
 giving $w = M^{\frac{1}{M}}$.

Comment: It is fun to write this line of reasoning as follows:

So we have shown:

If w is a value between $\frac{1}{e^e}$ and $e^{\frac{1}{e}}$, then the sequence w^{w^w} converges to some value M. And this value M is the input that produces the output w on the graph of $y = x^{\frac{1}{x}}$.



This resolves the paradox of the popular "proof" allegedly establishing that 1 = 2. It goes as follows:

If
$$w^{w^w} = 2$$
, then
 $2 = w^{w^w} = w^{(w^w)} = w^2$, and so $w = \sqrt{2}$
Task 1 thus establishes that $\sqrt{2}^{\sqrt{2^{\sqrt{2}}}} = 2$.

Task 2: Solve
$$w^{w^{w^*}} = 4$$
.

Task 1: Solve $w^{w^w} = 2$.

If $w^{w^{w^{*}}} = 4$, then $4 = w^{w^{w^{*}}} = w^{(w^{w^{*}})} = w^{4}$, and so $w = \sqrt{2}$. Task 2 thus establishes that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = 4$.

Thus both 2 and 4 equal $\sqrt{2}^{\sqrt{2^{\sqrt{2^*}}}}$ and so 2 = 4 or, equivalently, 1 = 2.

RESEARCH CORNER

Are there negative number solutions to $x^y = y^x$? How does the graph of this equation appear in all four quadrants?

Describe the complex number solutions to $x^y = y^x$. For example, x = i, y = -i is a solution. (We have

$$i^{-i} = \left(e^{i\frac{\pi}{2}}\right)^{-i} = e^{\frac{\pi}{2}} = \left(e^{-i\frac{\pi}{2}}\right)^{i} = (-i)^{i}.$$

APPENDIX: A swift tricky proof!

We claimed that if x and y are positive rational solutions to $x^{\frac{1}{x}} = y^{\frac{1}{y}}$ with x < y, then we must have $x = \left(\frac{n+1}{n}\right)^n$ and $y = \left(\frac{n+1}{n}\right)^{n+1}$ for some positive integer n. Let's prove this.

Assume x and y are rationals with 0 < x < y satisfying $x^{\frac{1}{x}} = y^{\frac{1}{y}}$.

We can write y = rx for some number r > 1. Since r = y / x, it too is rational.

From
$$x^{\frac{1}{x}} = (rx)^{\frac{1}{rx}}$$
 we get

$$\frac{\ln x}{x} = \frac{\ln r + \ln x}{rx}$$

giving

$$\ln x = \frac{\ln r}{r-1} = \ln \left(r^{\frac{1}{r-1}} \right).$$

Hence

$$x = r^{\frac{1}{r-1}}$$
$$y = rx = r^{\frac{r}{r-1}}.$$

Since r > 1 we can write $r = 1 + \frac{m}{n}$ for some rational number $\frac{m}{n}$. (Let's assume here m and n are positive integers with no

common factor different from $1.) \ \mbox{Thus we} \ have$

$$x = \left(\frac{n+m}{n}\right)^{\frac{n}{m}}$$
$$y = \left(\frac{n+m}{n}\right)^{\frac{n}{m}+1}$$

We need to prove that m = 1.

We are assuming that x is a rational

number. So let's write $x = \frac{a}{b}$, a reduced

fraction. Then we have $\left(\frac{n+m}{n}\right)^{\frac{n}{m}} = \frac{a}{b}$, or $\left(\frac{n+m}{n}\right)^n = \left(\frac{a}{b}\right)^m$.

Since *n* and *m* share no common factor other than 1, $\frac{n+m}{n}$ is also a reduced fraction.

Actually
$$rac{\left(n+m
ight)^n}{n^n}$$
 and $rac{a^m}{b^m}$ reduced

fractions too, and they are equal. We must have then that

$$\left(n+m\right)^n=a^m$$

and

$$n^n = b^m$$

Look at the second equation. If *n* has prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ and *b* has

prime factorization $b = p_1^{\ \beta_1} p_2^{\ \beta_2} \cdots$, then we have

$$p_1^{n\alpha_1} p_2^{n\alpha_2} \cdots = p_1^{m\beta_1} p_2^{m\beta_2} \cdots.$$

From this it follows that

 $n\alpha_i = m\beta_i$

for each i. Since n and m have no common factors, it follows that α_i is a multiple of m for each i. If we write $\alpha_i = mk_i$ we get that

$$n = p_1^{mk_1} p_2^{mk_2} \cdots = \left(p_1^{k_1} p_2^{k_2} \cdots \right)^m$$
, an m the power.

In the same way, the first equation gives that n + m equals an m th power too.

Thus we can write

$$n = c^m$$
$$n + m = (c + d)^r$$

for some positive integers c and d.

Now we get

$$m = \left(c + d\right)^m - c^m.$$

This is problematic if m is an integer greater than 1 because then

$$m = \binom{m}{1}c^{m-1}d + \binom{m}{2}c^{m-2}d^2 + \cdots + \binom{m}{m-1}cd^{m-1} + d^m$$

is a sum of m positive integers not all equal to 1.

So it must indeed be the case that m = 1, just as we hoped.