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CURIOUS MATHEMATICS FOR FUN AND JOY


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## THIS MONTH'S PUZZLER

We have that $2^{4}=4^{2}$. Is this the only example of a pair of distinct positive integers satisfying $a^{b}=b^{a}$ ? Are there rational solutions to this equation?

THE GRAPH OF $y=x^{\bar{x}}$ FOR $x>0$.
Using graphing software we see that the
equation $y=x^{x}$ produces the graph
shown, at least to the right of the vertical axis. (What happens to the left?)


The graph appears to have a maximum value of about 1.4 somewhere near $x=2.7$, and decreases towards the value 1 as $x$ grows. It also looks as though it "wants to" adopt the value 0 at $x=0$.

With the aid of calculus one can prove that the maximal value of the graph is actually $\underline{1}$ $e^{\bar{e}} \approx 1.444$ occurring at $x=e \approx 2.71828$, and that, indeed, $\lim _{x \rightarrow 0^{+}} x^{\frac{1}{x}}=0$ and $\lim _{x \rightarrow \infty} x^{\frac{1}{x}}=1$. One can also prove that the graph is increasing on the interval $(0, e)$ and decreasing on $(e, \infty)$, as the picture suggests. This means, that for any value $M$ between 1 and $e^{\frac{1}{e}}$, there are precisely two values $a<e<b$ with
$a^{\frac{1}{a}}=M=b^{\frac{1}{b}}$.


One checks that $2^{\frac{1}{2}}=4^{\frac{1}{4}}$ and, of course, that $1^{\frac{1}{1}}=1^{\frac{1}{1}}$, and since 1 and 2 are the only positive integers below $e$ we have that $a=2$ and $b=4$ give the only distinct positive integer solution to the equation
$a^{\frac{1}{a}}=b^{\frac{1}{b}}$.

Are there distinct positive rational solutions to $a^{\frac{1}{a}}=b^{\frac{1}{b}}$ ?

Yes! A wee bit of algebra shows, for any positive integer $n, a=\left(\frac{n+1}{n}\right)^{n}$ and $b=\left(\frac{n+1}{n}\right)^{n+1}$ fit the bill.

Comment: Since the first term is clearly less than the second term, we must have that
$a=\left(1+\frac{1}{n}\right)^{n}<e$ and $b=\left(1+\frac{1}{n}\right)^{n+1}>e$.
In fact, as one learns in calculus class,
$\left(1+\frac{1}{n}\right)^{n}$ is an increasing sequence
converging to $e$. (So is $\left(1+\frac{1}{n}\right)^{n+1}$ a
decreasing sequence also converging to $e$ ?)

We will prove at the end of this essay that
Any rational solution to $a^{\frac{1}{a}}=b^{\frac{1}{b}}$ with $0<a<b$, must be of the form
$a=\left(\frac{n+1}{n}\right)^{n}$ and $b=\left(\frac{n+1}{n}\right)^{n+1}$ for some positive integer $n$.

## 

THE GRAPH OF $x^{y}=y^{x}$ FOR $x>0, y>0$.

If $a$ and $b$ are positive numbers satisfying
$a^{\frac{1}{a}}=b^{\frac{1}{b}}$, then "cross exponentiating" gives $a^{b}=b^{a}$. (And conversely, positive reals satisfying $a^{b}=b^{a}$ also satisfy $a^{\frac{1}{a}}=b^{\frac{1}{b}}$.)

As $a=2$ and $b=4$ are the only distinct positive integer solutions to $a^{\frac{1}{a}}=b^{\frac{1}{b}}$, they are unique distinct positive integer solutions to $a^{b}=b^{a}$ as well. This answers the opening puzzler.

The graph of the equation $x^{y}=y^{x}$, at least in the first quadrant, has two components: the diagonal line of points with equal coordinate values ( $x=y$ ) and a curve of points with distinct coordinate values.


As we have seen, for each positive integer $n$, the point $P_{n}=\left(\left(1+\frac{1}{n}\right)^{n},\left(1+\frac{1}{n}\right)^{n+1}\right)$ lies on the curve. As $n$ grows, these points approach $(e, e)$, which must thus be the point of intersection of the two components of the graph.

The result we prove at the end of this essay establishes that points $P_{n}$ and their reflections across the diagonal, are the only points on the graph off the diagonal with rational coordinates. (And the points $(2,4)$ and $(4,2)$ are the only off-diagonal points with integer coordinates.)

## 

## A CONNECTION TO $w^{w^{w}}$

Given a positive real number $w$, set $a_{1}=w$ and $a_{n+1}=w^{a_{n}}$ for each $n>1$.
This gives the sequence

$$
w, w^{w}, w^{\left(w^{w}\right)}, w^{\left(w^{\left(w^{w}\right)}\right)}, \ldots
$$

This sequence can converge to a finite value (it does for $w=1$, for instance) or grow without bound (say, for $w=2$ ).

Comment: When people write $w^{w^{w^{w}}}$ they mean the sequence $w, w^{w}, w^{\left(w^{w}\right)}, \ldots$ To say that $w^{w^{w "}}$ converges means that the sequence has a finite limit.

Swiss mathematician Leonard Euler (17071783) proved that the sequence converges for all values $w$ between $\frac{1}{e^{e}}$ (which is 1
about 0.066 ) and $e^{\bar{e}}$, which is the maximum value of the graph $y=x^{\frac{1}{x}}$ (which is about 1.444).

If $w$ is a value between $\frac{1}{e^{e}}$ and $e^{\frac{1}{e}}$, set $M$ to be the limit value of the sequence. Since $a_{n+1}=w^{a_{n}}$ and $a_{n+1}$ and $a_{n}$ each converge to $M$ as $n$ grows, we have

$$
M=w^{M}
$$

giving $w=M^{\frac{1}{M}}$.
Comment: It is fun to write this line of reasoning as follows:

$$
\begin{aligned}
& w^{w^{w^{w}}}=M \\
& w^{\left(w^{w^{m}}\right)}=M \\
& w^{M}=M \\
& w=M^{\frac{1}{M}}
\end{aligned}
$$

So we have shown:
If $w$ is a value between $\frac{1}{e^{e}}$ and $e^{\frac{1}{e}}$, then the sequence $w^{w^{w^{w}}}$ converges to some value $M$. And this value $M$ is the input that produces the output $w$ on the graph of $y=x^{\frac{1}{x}}$.

As my colleague Tim Pettus recently pointed out, if $M$ is a value between 1 and 1 $e^{e}$, then there are two values $w=a$ and $w=b$, for which $w^{w^{w^{w}}}$ converge to $M$.


This resolves the paradox of the popular "proof" allegedly establishing that $1=2$. It goes as follows:

Task 1: Solve $w^{w^{w}}=2$.

If $w^{w^{w}}=2$, then
$2=w^{w^{w}}=w^{\left(w^{w}\right)}=w^{2}$, and so $w=\sqrt{2}$.
Task 1 thus establishes that $\sqrt{2}^{\sqrt{2} \sqrt{2}}=2$.

Task 2: Solve $w^{w^{w}}=4$.

If $w^{w^{w}}=4$, then
$4=w^{w^{w}}=w^{\left(w^{w}\right)}=w^{4}$, and so $w=\sqrt{2}$.


Thus both 2 and 4 equal $\sqrt{2}^{\sqrt{2} \sqrt{\sqrt{2}}}$ and so $2=4$ or, equivalently, $1=2$.

##  RESEARCH CORNER

Are there negative number solutions to $x^{y}=y^{x}$ ? How does the graph of this equation appear in all four quadrants?

Describe the complex number solutions to $x^{y}=y^{x}$. For example, $x=i, y=-i$ is a solution. (We have

$$
\left.i^{-i}=\left(e^{i \frac{\pi}{2}}\right)^{-i}=e^{\frac{\pi}{2}}=\left(e^{-i \frac{\pi}{2}}\right)^{i}=(-i)^{i} .\right)
$$

##  APPENDIX: A swift tricky proof!

We claimed that if $x$ and $y$ are positive rational solutions to $x^{\frac{1}{x}}=y^{\frac{1}{y}}$ with $x<y$, then we must have $x=\left(\frac{n+1}{n}\right)^{n}$ and $y=\left(\frac{n+1}{n}\right)^{n+1}$ for some positive integer $n$. Let's prove this.

Assume $x$ and $y$ are rationals with $0<x<y$ satisfying $x^{\frac{1}{x}}=y^{\frac{1}{y}}$.

We can write $y=r x$ for some number $r>1$. Since $r=y / \mathrm{x}$, it too is rational. From $x^{\frac{1}{x}}=(r x)^{\frac{1}{r x}}$ we get

$$
\frac{\ln x}{x}=\frac{\ln r+\ln x}{r x}
$$

giving

$$
\ln x=\frac{\ln r}{r-1}=\ln \left(r^{\frac{1}{r-1}}\right) .
$$

Hence

$$
\begin{aligned}
& x=r^{\frac{1}{r-1}} \\
& y=r x=r^{\frac{r}{r-1}} .
\end{aligned}
$$

Since $r>1$ we can write $r=1+\frac{m}{n}$ for some rational number $\frac{m}{n}$. (Let's assume here $m$ and $n$ are positive integers with no common factor different from 1.) Thus we have

$$
\begin{aligned}
& x=\left(\frac{n+m}{n}\right)^{\frac{n}{m}} \\
& y=\left(\frac{n+m}{n}\right)^{\frac{n}{m}+1}
\end{aligned}
$$

We need to prove that $m=1$.

We are assuming that $x$ is a rational number. So let's write $x=\frac{a}{b}$, a reduced fraction. Then we have $\left(\frac{n+m}{n}\right)^{\frac{n}{m}}=\frac{a}{b}$, or

$$
\left(\frac{n+m}{n}\right)^{n}=\left(\frac{a}{b}\right)^{m}
$$

Since $n$ and $m$ share no common factor other than $1, \frac{n+m}{n}$ is also a reduced fraction.

Actually $\frac{(n+m)^{n}}{n^{n}}$ and $\frac{a^{m}}{b^{m}}$ reduced fractions too, and they are equal. We must have then that

$$
(n+m)^{n}=a^{m}
$$

and

$$
n^{n}=b^{m}
$$

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Look at the second equation. If $n$ has prime factorization $n=p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}} \cdots$ and $b$ has

