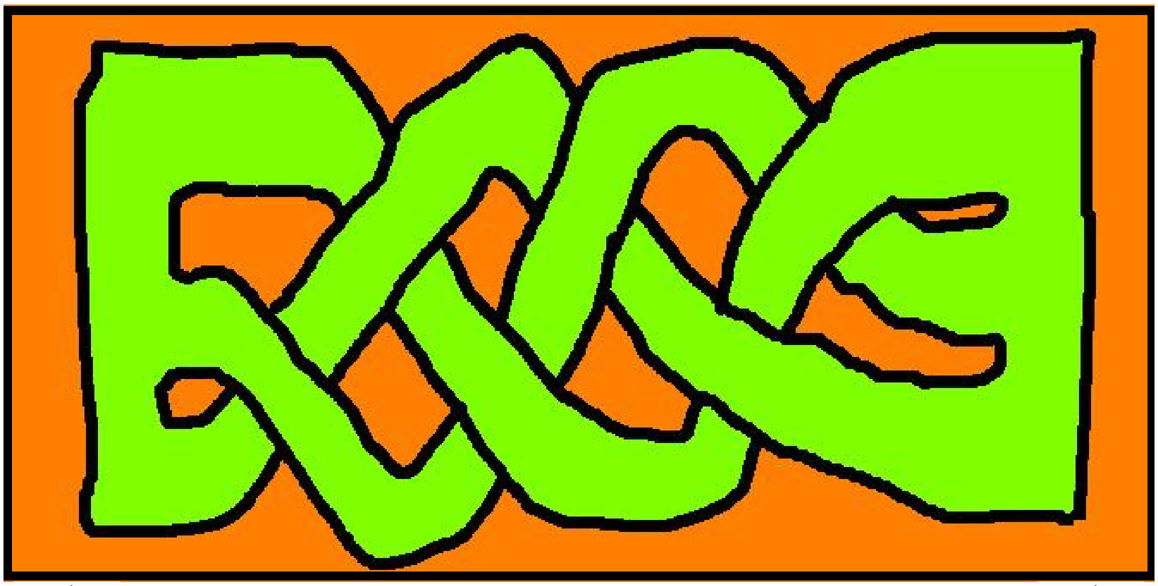

WILDLY COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

The third of a new Math Letter series!

JUNE 2012

Question: When should I stop using the adjective “new” in the subtitle above?

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

HOW TO THINK LIKE A SCHOOL MATH GENIUS: A Teacher’s Guide! I was recently invited to the SUM conference in Saskatoon. A fabulous group of folk thinking hard about good mathematics for their Province’s kids. Here is a lecture I gave there on genius thinking. (Okay .. self promotion!)
www.jamestanton.com/?p=1097

MATH FOR LOVE: Mathematicians and teachers Katherine Cook and Dan Finkel are doing spectacular work promoting super math thinking. Check out their website www.mathforlove.com.

CUT THE KNOT: Alex Bogomolny’s website www.cut-the-knot.org is intensely rich and chock-o-block full of incredible mathematical tidbits. You will be lost there for weeks. A top-notch resource by a top-notch mathematician.

A PUZZLER: Here is a classic puzzler recently told to me by Avery Pickford. As more promotion, check out his super mathematics blog: "Without geometry, life is pointless:" withoutgeometry.com.

DRAGONS and POISONS

You and a dragon have agreed to take part in the following "game." (I am not sure why but, well, that is how it is). At noon today you will each bring to the local coffee shop a goblet of poison. The dragon will take a sip of poison from your goblet and then a sip from his own. You will take a sip of poison from the dragon's goblet and then a sip from your own. You will then each sit and wait for the results.

Let me tell you about the poison.

There is only one type of poison available to each of you and it comes in varying strengths of potency. A single sip of any potency is enough for quite a detrimental effect (namely your or the dragon's complete demise) but it will take a few hours to act. There is an antidote to the poison: a sip of a stronger dose of the poison. Taking two sips – one dose followed by a stronger dose – has the same effect as not taking any poison at all. However, taking a second dose of equal or weaker potency will not help your predicament one whit.

Let me tell you something about the dragon: She has access to the most potent strength of poison of all (and you don't!)

So here is the challenge: *Given this knowledge of how poison works and the fact that the dragon might bring the most potent sample of all, is there a means for you to survive this cheery game?*

While we are at it, here are three more classic conundrums on matters of intention and action. (These puzzles truly are classic and have been bandied about for decades.)

UNEXPECTED QUIZ: Your teacher announces that he will give a math quiz one week-day next week during class, but that you will not at any time in advance know on which day the quiz shall occur. He says it will, for certain, come as a complete and utter surprise.

You then reason as follows:

Well … The quiz shan't be on Friday. For if it were coming on a Friday I would reason Thursday night, having not yet had the quiz, that it must be set for the next day and so won't come as a complete surprise.

Now with Friday ruled out, I can argue that the quiz can't occur on Thursday - for on Wednesday night, with no quiz on the first three days (and Friday not a consideration) I would expect a quiz the next day and so not be surprised.

In the same way there can be no quiz on Wednesday having ruled out Friday and Thursday. And having ruled out Wednesday, Thursday and Friday out goes Tuesday by the same reasoning.

The quiz must be Monday – but now that I know that, it isn't a surprise.

There is no day for the quiz to occur!

On Thursday you have a quiz and you are surprised!

This seems like a paradox. Is there a way to resolve this conundrum?

TWO ENVELOPES: I hand you two envelopes and tell you they each contain cash, one twice the amount of the other. You may pick an envelope, open it and count the cash.

I now give you a choice: Either keep that cash currently in hand, or give it back to me and keep instead the contents of the other envelope (without first looking inside). Given this choice, would you like to "stick" or "switch"?

Here's the paradox of this situation: Without further knowledge of the contents of the second envelope you must reason as follows: *The second envelope either contains double the cash or half the cash. There is a 50% chance by switching that I will double my money currently in hand, and a 50% chance of losing only half. Since I stand to gain more than I will lose, the odds are in my favour to switch. I'll switch!*

Now having reasoned this, I am going to ask: *Why didn't you just choose the second envelope in the first place?* In fact, no matter what you see in the first envelope you will always reason it is better to switch envelopes! Why bother opening the first envelope at all?

A SURPISE PRESENT: Harriotte hands you a wooden box and says:

"This box contains a present for you! You may open the box at noon and see what the present is. But before then you will have absolutely no idea what lies in the box. And when you do open the box, its contents will truly be a complete surprise to you. The box contains nine plastic turtles and that is your present."

At noon, should you be surprised?

TACKLING THE CONNUNDRUMS:

Dragons and Poisons: Here is the standard answer to the puzzle:

The dragon has nothing to gain by bringing anything but the strongest poison available. She will sip your poison and then be cured from it by sipping her own. You, on the other hand, will sip the strongest poison first and will not be cured by a sip of your own since it is sure to be weaker. Your doom seems guaranteed.

BUT NOT QUITE … You know that the dragon will reason to bring the strongest poison. Use it as a cure for you!

SOLUTION: Sip some poison just before the game and bring a goblet of water to the coffee house. You will be cured by the dragon's poison AND the dragon will be killed by her own poison (since you brought only water!)

Dragon …	Bring strongest poison
You …	Bring water and sip poison beforehand
Result:	
Dragon= DEMISE	
You = SURVIVAL	

This sounds grand, but let's not under-estimate the intelligence of dragons! She knows that you will reason she will bring the strongest poison and that you will operate in accordance – bringing water to the coffee-house and drinking a dose of poison just before the game. She can foil your move by actually bringing water to the coffee house! You will be doomed and she will survive! (Another option: The dragon could, instead, bring the strongest poison to the table and sip some weak poison beforehand like you. This will lead to survival for each of you. But witnessing your demise seems more dragon-like. She will opt for that.)

Dragon …	Bring water	
You …	Bring water and sip poison beforehand	
Result:	Dragon	= SURVIVAL
	You	= DEMISE

HOWEVER … You too are no fool, and will reason that the dragon will reason that this is how you will reason about how she reasons! You can foil this dragon's change of strategy by not drinking anything at all beforehand (and you will both drink water and both survive) OR, better yet, do drink a mild dose of poison before the game and bring along a stronger dose! You will survive and the dragon will not.

But the dragon knows you will operate this way … AND SO ON!

This game actually has no stable solution if both players are rational thinkers: For any set of choices each player may make (bring poison or bring water, sip poison beforehand or don't) there is reason for at least one of the players to change strategy. I don't have any particular advice for you if you are ever invited to play such a game with a dragon!

Experiment: It would be interesting to test human psychology and play a non-lethal version of the game. Cards from a deck can represent doses of poison. Each of two players can select a card, two through king, and bring it to the table - and perhaps slip a second card in her pocket to represent a dose of poison ingested beforehand. The person playing the dragon has the option to bring an ace card, trumping all other values.

Have a class of students play the game. What seems to be the most common human strategy for the non-dragon? For the dragon? How do strategies change if players play against each other multiple times and learn each others' thinking?

A Surprise Present: To resolve this paradox ask: *Can you trust anything Harriotte said to you?*

The answer must be "no" since she is contradicted herself by saying that the contents will be complete surprise and, at the same time, by telling you what the contents supposedly are. Logically, you must reject everything Harriotte said about the contents of the box and your reaction to them. Even if at noon you do find nine plastic turtles inside you have no logical reason to expect this, and so will be surprised!

By the way … If there are indeed turtles in the box, then everything Harriotte said to you is consistent and not at all contradictory from her perspective: She knew you would have to reject all that she said and so possess no knowledge of the contents of the box AND she knew that she did give you nine turtles! She did not lie. You simply had no means to know that she didn't!

The Unexpected Quiz follows the same rationale. The line of reasoning leading you to conclude that there can be no day for a surprise quiz is valid and shows that you cannot believe what the instructor said! So, in fact, you possess no information about the day of the quiz (if there is a quiz at all). And if one were to come one day of next week, it will indeed be a logical surprise!

And again…There is no contradiction from the instructor's point of view!

Two Envelopes: Here we should question here the 50-50 rationale: *Would you always believe there is a 50% chance the second envelope contains double the cash?* No!

If you found a single penny in the first envelope, for example, then you would know for certain that the other contains double the cash. (There is no such thing as half a penny.) Here there is 100% incentive to "switch."

At the other end of the spectrum, if you found a huge amount of cash in the first envelope you might well reason that is very unlikely there be even more cash in the second. The incentive to "stick" is strongest here.

One resolution of the paradox is to simply reject the 50/50 rationale presented. Fair enough. But perhaps the following extension is perturbing:

TWO-ENVELOPE EXTENSION:
You will set up the two-envelope game for a friend to play – placing cash into two envelopes, one double the amount the other. But you are charged to do this in a way so that there could NEVER be a chance your friend will, for certain, know that switching is best.

Can you set up such a game?

For instance, you will never place one penny in one envelope and two in the other. For, if by chance, your friend were to pick the one-cent envelope first she would know for sure to switch.
(By the same reasoning you would never set up a "n-cent/$2n$-cent" game with n odd.)

But you would also never set a 2-cent/4-cent game. For, if by chance, your friend picked the two-cent envelope she would know for sure to switch because she realises you would never set up a 1-cent/2-cent game. (And by the same reasoning all $2n$-cent/$4n$-cent games, n odd, are out.)
And the 4-cent/8-cent game is out too For if by chance, your friend were to open the 4-cent envelope she would know for sure to switch as you would never set up and 2-cent/4-cent game.

AND SO ON!

This reasoning rules out **all** possible choices of numbers! Your friend, knowing your thinking, will always know to switch!

Question: Your brain hurts so you just go ahead and put $1.28 in one envelope and double in the other. What happens?

TOUGH QUESTION: Suppose cash is infinitely divisible and in grand abundance: half pennies exist, quarter pennies exist, and so on. And a million pennies in an envelope is not considered unlikely, nor a billion or a quadrillion pennies. Is there a means to resolve this infinite, "continuous" version of things?

Research Corner:
BACK TO DRAGONS

Given the lack of stable solution to the game and the inherent randomness in what each player will settle to do, game theorists would advise you and the dragon each employ a "mixed strategy" and maximize any advantage randomness could offer. You each have two choices to make:

1) Bring poison or bring water
2) Sip weak poison beforehand or not.

[You might argue it behooves the dragon to bring the strongest dose of poison if she brings any at all.] A mixed strategy has you each flip a coin to decide what to do, but make it a biased coin. Let d_1 be the probability that the dragon will choose to bring poison, and d_2 the probability she will sip poison beforehand. Let y_1 and y_2 represent your two corresponding probabilities. Are there values for d_1, d_2, y_1, y_2 that simultaneously maximize each of your chances of survival?

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