MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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Workshop 6: Charging a Capacitor: Solutions

Group ____________________________________

Names ____________________________________

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Introduction:

In our ***Workshop 5: Analyzing DC Circuits*** we used the two circuit laws:

Current Conservation: At any point where there is a junction between various current carrying branches, the sum of the currents into the node must equal the sum of the currents out of the node.

$$I_{in} = I_{out}$$

Loop Rule: The sum of the voltage drops ΔV_i, across any circuit elements that form a closed circuit is zero.

$$\sum_{i=1}^{i=N} \Delta V_i = 0 .$$

We also needed conventions for describing the

- direction of the current in any branch
- direction for circulation around a closed circuit
- definition of the voltage difference across a circuit element.

We shall now add to our list of circuit conventions, by defining the convention for the voltage difference across a capacitor.

Capacitors:

Consider the following circuit containing an electromotive source ε, a resistor R, a capacitor C, and a switch S.

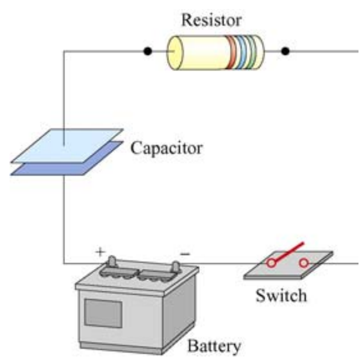


Figure 1: RC-circuit

Question 1: When the switch is closed, choose a direction for positive current and a direction for circulation. Indicate your choices in Figure 1.

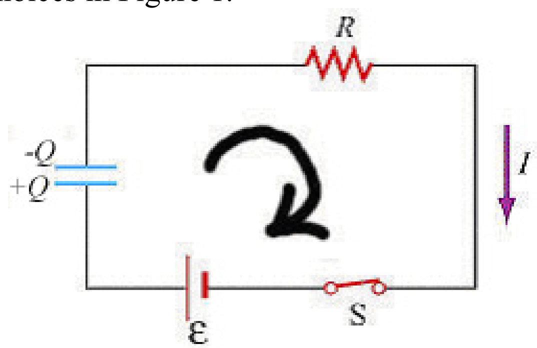


Figure 1: Choices for current, circulation, and charge on capacitor

We now need to introduce our conventions for determining the voltage drop across the capacitor. Think of the capacitor as consisting of two separate conducting surfaces that have equal and opposite charges. So we must choose which plate has positive charge, $+Q$ and which plate has negative charge, $-Q$ for the capacitor. Just as in the case for our choice of sign for current, if we solve for the charge Q, and discover that our result for Q is negative, then the plate we chose as positive is actually negative.

Question 2: Choose which of the capacitor plates in Figure 1 are positive and negative and draw their charges on Figure 1.

The choice of circulation defines what we mean by 'before' and 'after' the capacitor. The plate that is charged positively is at a higher voltage than the plate that is charged negatively.

Question 3: Suppose we choose clockwise circulation direction, current, and positive and negative charged plates as shown in Figure 2. What is the voltage difference across the capacitor plates?

Answer:

$$\Delta V \equiv V_{after} - V_{before} = +Q/C$$

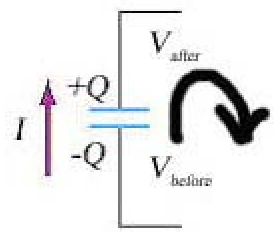


Figure 2: Choice of three conventions for capacitor, with circulation direction clockwise

Question 4: Suppose we choose counterclockwise for the direction of circulation, current, and positive and negative charged plates as shown in Figure 3. What is the voltage difference across the capacitor plates?

Answer:

$$\Delta V \equiv V_{after} - V_{before} = -Q/C$$

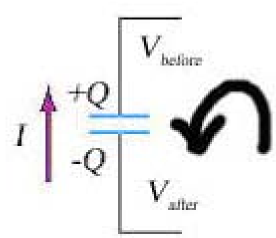


Figure 3: Choice of circulation direction counterclockwise

Relationship between the charge Q and the current I :

Current is defined to be the flow of charge, and by flow we mean the rate of change of charge in time. So the temptation is to always assume that $I = +dQ/dt$. However we must take into account our choice of signs for positive current and positive charge. Is $I = \pm dQ/dt$? Notice that this relation between charge and current does not depend on our choice of direction for circulation. You may find it easier to imagine that the direction of current corresponds to the direction positive charges are flowing.

Question 5: Suppose we choose the direction of the current, I, to flow towards our choice of positive plate (Figure 4). Is $I = \pm dQ/dt$? Explain your reasoning.

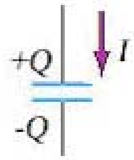


Figure 4: Charge-current convention

Answer:

Then from the Figure 6, we see that as time progresses, positive charges are building up on the +Q plate. The charge on the plate increases and $dQ/dt > 0$. Therefore in order for our signs to agree,

$$I = +dQ/dt.$$

Question 6: Now suppose we chose the direction of positive current to flow away from the positive plate (Figure 5). Is $I = \pm dQ/dt$? Explain your reasoning.

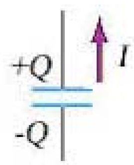


Figure 5: Charge-current convention

Answer:

The charge on the plate decreases and $dQ/dt < 0$. Remember that we treat I as positive, $I > 0$. Therefore in order for our signs to agree,

$$I = -dQ/dt.$$

Summary: The rules for determining the voltage differences across circuit elements and the two circuit laws are all the tools that you will need to analyze circuits involving voltage sources, resistors and capacitors.

Problem 1: Ch*arging a Capacitor*

Consider the circuit shown in Figure 6. The circuit consists of an electromotive source ε, a resistor R, a capacitor C, and a switch S.

Question 7: Choose a direction for the current, a direction for circulation around the closed loop, and the signs on the capacitor plates, and draw these on figure 6.

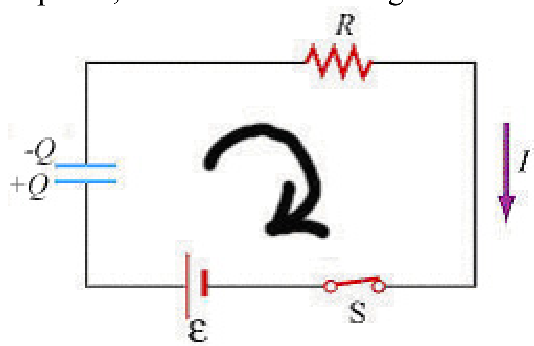


Figure 6: Choices for current and circulation

Question 8:

At $t = 0$, the switch S is closed in Figure 1. The capacitor initially is uncharged, $Q(t = 0) = 0$. What is the current that begins to flow in the circuit after a very short time has passed?

Answer:

Since the capacitor is uncharged, it has no effect on the circuit, acting like a short circuit. Therefore the current should take the initial value $I_0 = \mathcal{E}/R$.

Question 9: What is the current in the circuit after a very long time has passed?

Answer:

Eventually the voltage difference between the capacitor plates equals the voltage difference of the source. Then the electric field in the wire connecting the battery to the cpacitor approaches zero, so there is no longer an electric force to drive the current in the circuit. So after a very long time the current in the wire is nearly zero.

Question 10: Why will the current decrease as a function of time?

Answer:

As the charge builds up on the capacitor plates, the voltage difference between the capacitor plates increases, so the voltage difference between either terminal of the battery and the plate it is connected to decreases. So the electric field in the wire decreases. Therefore the current in the wire will decrease in time.

Question 11: Use the Loop Rule for the closed RC circuit shown in Figure 6 to find an equation involving the charge Q on the capacitor plate, the capacitance C, the current I in the loop, the electromotive source ε, and the resistance R.

Answer:

$$\varepsilon - \frac{Q}{C} - IR = 0$$

Question 12: What is the relation between the current in the circuit and the charge on the capacitor plate?

Answer:

The current is related to the charge on the capacitor by

$$I = +dQ/dt$$

Question 13: Use the results of Questions 11 and 12 to find the differential equation that describes how the charge $Q(t)$ on the positive capacitor plate varies in time. Your equation should include terms that involve Q, dQ/dt, R, and ε.

Answer:

So our differential equation for the circuit is

$$\varepsilon - \frac{Q}{C} - \frac{dQ}{dt}R = 0 .$$

Summary: ***Solving the Charging Differential equation for a Capacitor***

The charging capacitor satisfies a first order differential equation that relates the rate of change of charge to the charge on the capacitor:

$$\frac{dQ}{dt} = \frac{1}{R}\left(\varepsilon - \frac{Q}{C}\right)$$

This equation can be solved by the method of separation of variables. The first step is to separate terms involving charge and time, (this means putting terms involving, dQ and Q on one side of the equality sign and terms involving dt on the other side),

$$\frac{dQ}{(\varepsilon - \frac{Q}{C})} = \frac{1}{R}dt$$

or

$$\frac{dQ}{Q - C\varepsilon} = -\frac{1}{RC}dt$$

Now we can integrate both sides of the above equation,

$$\int_0^{Q(t)} \frac{dQ}{Q - C\varepsilon} = -\frac{1}{RC}\int_0^t dt$$

which yields

$$\ln\left(\frac{Q(t) - C\varepsilon}{-C\varepsilon}\right) = -\frac{t}{RC}$$

This can now be exponentiated using the fact that $\exp(\ln x) = x$ to yield

$$\boxed{Q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) = Q_f\left(1 - e^{-t/RC}\right)}$$

where $Q_f = C\varepsilon$ is the maximum amount of charge stored on the plates. The time dependence of $Q(t)$ is plotted in the Figure 7 below:

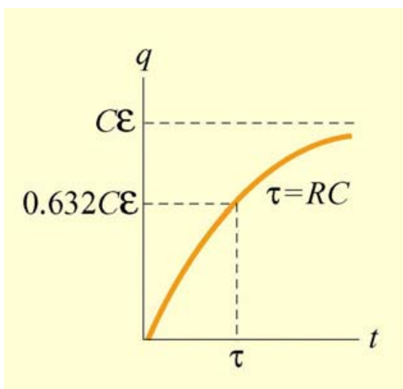


Figure 7: charge vs. time graph

Question 14: How does the voltage across the capacitor vary as a function of time?

Answer:

The voltage across the capacitor is given by

$$V_C = Q(t)/C = \varepsilon\left(1 - e^{-t/RC}\right).$$

So the voltage slowly increases until it reaches a final value equal to the voltage source ε.

Question 15: Why does the charge on the capacitor approach a constant value after a sufficiently long time has passed since the switch was closed?

Answer:

Since the voltage across the capacitor approaches the voltage across the terminals, the electric field in the wires approaches zero, and so the current approaches zero. Therefore no more charge will flow to or from the plates of the capacitor.

Time Constant:

The current that flows in the circuit is equal to the derivative in time of the charge,

$$I = \frac{dq}{dt} = \left(\frac{\varepsilon}{R}\right)e^{-t/RC} = I_0 e^{-t/RC}$$

This function is often written as $I(t) = I_0\, e^{-t/\tau}$ where $\tau = RC$ is called the *time constant.*

Question 16: Show that the units of $\tau = RC$ are seconds.

Answer:

Since resistance $R = V/I$, the units of resistance are

$$[ohms] = [volts/amps] = [volts - \mathrm{sec}/coulombs].$$

Also capacitance $C = Q/V$, the units of capacitance are

$$[farad] = [coulombs/volts].$$

Therefore the units of the time constant, $\tau = RC$, are

$$[volts - \mathrm{sec}/coulombs][coulombs/volts] = [\mathrm{sec}].$$

The time constant τ is a measure of the decay time for the exponential function. This decay rate satisfies the following property

$$I(t+\tau) = I(t)\,e^{-1}$$

i.e. after one time constant τ has elapsed, the current falls off by a factor of $e^{-1} = 0.368$, as indicated in Figure 8 .

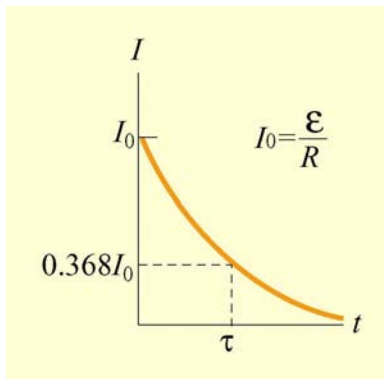


Figure 8: current vs. time graph

Problem 2: *Multiple Loop Circuits (Challenge Question)*

Consider the following circuit consisting of a voltage source ε_1, three resistors with resistances R_1, R_2, and R_3, and a capacitor with capacitance C connected together as shown in Figure 9.

Question 17: This circuit has three branches. Identify the branches in this multiloop circuit and choose positive directions for the flow of currents I_1, I_2, and I_3 in each branch. Draw the direction of your currents in Figure 9.

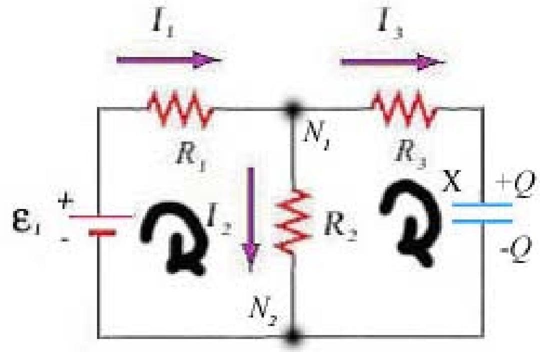


Figure 9: Multiloop current

Current conservation:

There are two node points in the circuit, N_1 and N_2, where current branches off (point N_1) or recombines (point N_2). (It helps to think of the flow of water in pipe that branches into two pipes and then recombines into one pipe). At each point the current into the node equals the current flowing out of the node,

$$I_{in} = I_{out}\,.$$

Question 18: Write down the equation for current conservation.

Answer:

$$I_1 = I_2 + I_3$$

Loop Rules:

There are three closed loops:

- Loop 1: formed by the voltage source ε_1 and the two resistors R_1 and R_2,
- Loop 2: formed by the capacitor C, and the two resistors R_3 and R_2,
- Loop 3: formed by the voltage source ε_1, the capacitor C, and the two resistors R_1 and R_3.

Loop 1 and Loop 2 are clearly visible. However, the outer perimeter of the circuit also forms Loop 3.

Question 19: Choose directions of circulations for each loop and use the Loop Law to write down equations for each loop describing the sum of the voltage differences around the closed loop.

Answer:

Loop 1: $\varepsilon_1 - I_1 R_1 - I_2 R_2 = 0$

Loop 2: $-\dfrac{Q}{C} + I_2 R_2 - I_3 R_3 = 0$

Loop 3: $\varepsilon_1 - I_1 R_1 - I_3 R_3 - \dfrac{Q}{C} = 0$

By either adding (or subtracting) the loop equations for Loop 1 and Loop 2 (depending on your choice of circulation direction), the voltage difference across resistor R_2 for these two loops have opposite (or the same) signs. Hence when we add (or subtract) the two equations, the voltage difference across resistor R_2 cancel, leaving the Loop Rule for Loop 3 as the result. Thus even though there are three loops, there are only two independent equations.

Question 20: In order to find the differential equations that describe this multi-loop circuit, find a relationship between the charge on the capacitor and the current that flows in that branch.

Answer:

In branch 3, the current that charges the capacitor is I_3, therefore

$$I_3 = \frac{dQ}{dt}$$

Question 21: Using your results from Question 18-20, write down the two Loop equations as differential equations for the charge on the capacitor plate Your equations should include terms that involve Q, dQ/dt, R_1, R_2, and R_3, and $\boldsymbol{\varepsilon}_1$.

Answer:

Current conservation: $I_1 = I_2 + \dfrac{dQ}{dt}$

Loop 1: $\varepsilon_1 - I_1R_1 - I_2R_2 = 0$

Loop 2: $-\dfrac{Q}{C} + I_2R_2 - \dfrac{dQ}{dt}R_3 = 0$

Question 22: At $t = 0$, what is the voltage difference across the capacitor?

Answer:

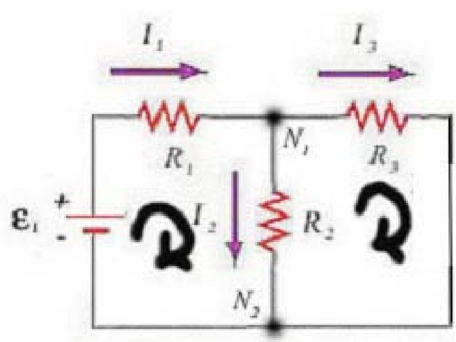
Once again the capacitor is uncharged so the voltage difference across the capacitor is zero.

$$V_C = \frac{Q(t=0)}{C} = 0\,.$$

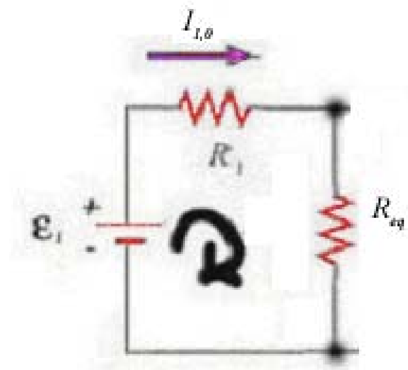
Question 23: Based on your result from Question 22, find the current that flows in the branch containing the voltage source at $t = 0$.

Answer:

Since the capacitor acts like a short circuit the circuit looks like



This circuit can be easily solved by reducing the two resistors, R_2, and R_3, that are in parallel, to an equivalent resistor $R_{eq} = R_2R_3/R_2 + R_3$. Then the circuit looks like



The current from the voltage source can now be easily determined,

$$I_{1,0} = \frac{\varepsilon_1}{R_1 + R_{eq}} = \frac{\varepsilon_1 (R_2 + R_3)}{R_1 (R_2 + R_3) + R_2 R_3}.$$

Question 24: When a long time has passed after the switch was closed, what is the current that flows in the branch of the circuit that included the capacitor?

Answer:

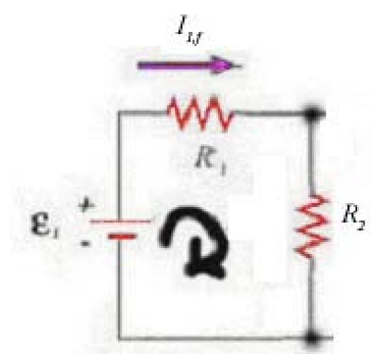
Now the capacitor is fully charge so no current flows in branch 3, hence the capacitor acts like an open circuit and $I_3 = 0$.

Question 25: When a long time has passed after the switch was closed (based on your result form Question 24), find the current that flows from the voltage source?

Answer:

When a long time has passed after the switch was closed (based on your result form Question 8), find the current that flows from the voltage source?

Answer: The circuit looks like



The current in the circuit is then

$$I_{1,f} = \frac{\varepsilon_1}{R_1 + R_2}$$

Question 26: (Hard) Try to combine your two loop equations and current conservation, to find a single differential equation describing the rate of change of the charge on the capacitor plate. Your equations should include terms that involve Q, dQ/dt, R_1, R_2, and R_3, and $\boldsymbol{\varepsilon}_1$.

Answer:

Using the loop 1 equation from question 21, $\varepsilon_1 - I_1 R_1 - I_2 R_2 = 0$**,** we can solve for the current I_1 in terms of R_1, R_2, I_2, and ε_1,

$$I_1 = \frac{\varepsilon_1 - I_2 R_2}{R_1}.$$

We can use our above result for I_1 in the current conservation equation from question 21, $I_1 = I_2 + \dfrac{dQ}{dt}$, and solve for I_2,

$$I_2 = \frac{(\varepsilon_1/R_1 - dQ/dt)}{(1 + R_2/R_1)}.$$

We can now use loop 2 equation from question 21, $-\dfrac{Q}{C} + I_2 R_2 - \dfrac{dQ}{dt} R_3 = 0$, and our above result for I_2, to derive the differential equation for the charge on the capacitor

$$-\frac{dQ}{dt}\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) - \frac{Q}{C} + \varepsilon_1\left(\frac{R_2}{R_1 + R_2}\right) = 0.$$

Question 27: (Hard) Based on your result from Question 26, try to determine the time constant for this circuit? You do not have to solve your equation. See if you can 'read off' the time constant based on the comparison between your equation and the charging equation for a single loop RC circuit.

Answer:

We can rewrite this equation as

$$\frac{dQ}{dt} = +\varepsilon_1\left(\frac{R_2}{R_3 R_1 + R_3 R_2 + R_1 R_2}\right) - \frac{Q(R_1 + R_2)}{C\left(R_3(R_1 + R_2) + R_1 R_2\right)}.$$

When we compare this to our equation for the simple RC circuit,

$$\frac{dQ}{dt} = \frac{1}{R}\left(\varepsilon - \frac{Q}{C} \right).$$

We can set

$$R \equiv \frac{R_3\left(R_1 + R_2\right) + R_1 R_2}{R_1 + R_2}$$

and

$$\frac{\varepsilon}{R} \equiv \varepsilon_1 \left(\frac{R_2}{R_3 R_1 + R_3 R_2 + R_1 R_2} \right) .$$

This implies that

$$\varepsilon = \varepsilon_1 \left(\frac{R_2}{R_1 + R_2} \right).$$

We can then use our standard solution to the differential equation for the charge on the capacitor, with the above values for ε and R,

$$Q(t) = C\varepsilon\left(1 - e^{-t/RC}\right).$$

The time constant is then

$$\tau = RC = \left(\frac{R_3\left(R_1 + R_2\right) + R_1 R_2}{R_1 + R_2} \right) C .$$